On integrability of the Kontsevich non-abelian ODE system

Thomas Wolf
Brock University (Ontario, Canada)
twolf@brocku.ca
tel (1) 905 688 5550 x 3803, fax (1) 905 378 5713

Olga Efimovskaya
Moscow State University (Russia),
olga.efimovskaya@gmail.com

April 23, 2011

Abstract

We consider systems of ODEs with the right hand side being Laurent polynomials in several non-commutative unknowns. In particular, these unknowns could be matrices of arbitrary size. An important example of such a system was proposed by M. Kontsevich. We prove the integrability of the Kontsevich system by finding a Lax pair, corresponding first integrals and commuting flows. We also provide a pre-Hamiltonian operator which maps gradients of integrals for the Kontsevich system to symmetries.

1 Introduction

In connection with the theory of non-commutative elliptic functions, M. Kontsevich [1] considered the following discrete map

$$u \to uvu^{-1}, \qquad v \to u^{-1} + v^{-1}u^{-1}, \tag{1.1}$$

where u, v are non-commutative variables (in particular, $n \times n$-matrices of arbitrary size). His numerical computer experiments have shown that this map could be integrable (see [2]). In the abelian case the element

$$h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1} \tag{1.2}$$

is an integral for the mapping (1.1). The equation $h = const$ defines a family of elliptic curves. In the non-abelian case the element h is transformed as $\bar{h} = uhu^{-1}$. It follows from this formula that $trace(h^k)$ is a first integral of (1.1) for any natural k.

Kontsevich also observed that (1.1) is a discrete symmetry of the following non-abelian ODE system:

$$u_t = uv - uv^{-1} - v^{-1}, \qquad v_t = -vu + vu^{-1} + u^{-1} \tag{1.3}$$

and conjectured that (1.3) is integrable itself.

Our paper is devoted to the system (1.3). It belongs to the class of systems of the form

$$u_t = P_1(u, v), \qquad v_t = P_2(u, v), \tag{1.4}$$

where P_i are elements of the associative algebra $\mathcal{M}$ of all non-commutative polynomials in u, v, u^{-1}, v^{-1} with constant scalar coefficients. The elements of $\mathcal{M}$ are called *non-abelian Laurent polynomials.*

In papers [3, 4] integrable systems of type (1.4) with P_i being non-abelian polynomials in u, v, were considered. The existence of an infinite series of infinitesimal symmetries was taken as a criterion for integrability. A similar approach to integrability of evolutionary polynomial non-abelian PDEs was developed in [5]. As far as we know integrable systems with non-abelian Laurent right hand sides were not considered before.

In this paper we find a Lax representation with a spectral parameter for system (1.3). The corresponding Lax L-operator generates infinitely many integrals of motion for (1.3). They are integrals for the discrete map (1.1) as well. We also find a pre-Hamiltonian operator that maps gradients of first integrals to symmetries. This proves that (1.3) is integrable in the sense of [3].

2 Symmetries and integrals

Let us briefly recall the definitions from [3] generalized to the Laurent case.

Let $x_1, \ldots, x_N$ be non-commutative variables. We consider ODE systems of the form

$$\frac{dx_\alpha}{dt} = F_\alpha(x_1, ..., x_N, x_1^{-1}, ..., x_N^{-1}), \tag{2.5}$$

where F_α are Laurent polynomials. We denote by $\mathcal{M}$ the associative algebra of all Laurent polynomials. Formulas (2.5) together with $\frac{d(x_\alpha^{-1})}{dt} = -x_\alpha^{-1}\frac{dx_\alpha}{dt}x_\alpha^{-1}$ define the corresponding derivation D_t on $\mathcal{M}$.

An (infinitesimal) symmetry for (2.5) is a system

$$\frac{dx_\alpha}{d\tau} = G_\alpha(x_1, ..., x_N, x_1^{-1}, ..., x_N^{-1}), \qquad G_\alpha \in \mathcal{M} \tag{2.6}$$

compatible with (2.5). Compatibility means that the derivations D_t and D_τ corresponding to (2.5) and (2.6) commute.

The existence of an infinite series of symmetries is typical for integrable non-abelian ODEs. The simplest non-trivial symmetry for (1.3) is given by

$$\begin{aligned} u_\tau &= -uvu - uv^2 + uv + (vu)^{-1} + v^{-2} + u^2v^{-1} - uvu^{-1} + uv^{-2} + (vuv)^{-1} + u(vuv)^{-1}, \\ v_\tau &= \quad vuv + vu^2 - vu - (uv)^{-1} - u^{-2} - v^2u^{-1} + vuv^{-1} - vu^{-2} - (uvu)^{-1} - v(uvu)^{-1}. \end{aligned}$$

The following conjecture is concerned with the dimensions of vector spaces S_k of all symmetries with right hand sides being polynomials in u, v, u^{-1}, v^{-1} of degree $\le k$.

Conjecture:
$\dim S_{4n} = 2n^2, \quad \dim S_{4n+1} = 2n^2+2n, \quad \dim S_{4n+2} = 2n^2+2n+1, \quad \dim S_{4n+3} = 2n^2+4n+1.$

We have verified the conjecture for S_k, $k = 0, ..., 16$. Notice that (1.3) is invariant with respect to involution

$$u \to v, \qquad v \to u \tag{2.7}$$

and $t \to -t$. All known symmetries are either involution invariant up to $\tau \to -\tau$ or involution symmetric to another symmetry. In the remainder of the paper we will denote the involution $F|_{v\leftrightarrow u}$ of any Laurent polynomial F by $\overline{F}$.

An element I of $\mathcal{M}$ is called a *full integral* for (2.5) if $D_t(I) = 0$. In the matrix case this means that every component of I is an integral of motion. For equation (1.3) the element

$$I = uvu^{-1}v^{-1} \tag{2.8}$$

is a full integral. This element is an integral for the mapping (1.1) as well. We were unsuccessful in finding more Laurent full integrals for (1.3) (other than polynomials of I, I^{-1}).

In the q-case, which is a specialization of our general non-abelian situation, an additional full integral exists. Namely, consider the associative algebra $\mathcal{Q}$ generated by u, v with the relation $u\,v = q\,v\,u$, where q is a fixed constant. In this case the element

$$J = u + qv + qu^{-1} + v^{-1} + u^{-1}v^{-1}$$

is a full integral for (1.3). This integral is a q-deformation of the element (1.2).

Apart from (2.8), equation (1.3) has so called trace first integrals. If u and v are matrices, these integrals are given by traces of some Laurent polynomials. For instance, the traces of h^k, where h is given by (1.2) and $k \in \mathbb{N}$, are such integrals.

For (2.5) with x_α being non-commutative symbols we define the $trace(a)$, $a \in \mathcal{M}$ as an equivalence class. Two elements a and b of $\mathcal{M}$ are called equivalent iff a can be obtained from b by cyclic permutations of factors in its monomials. In other words, the traces for elements of $\mathcal{M}$ are defined as the corresponding elements of the quotient space $\mathcal{M}/K$, where K is the vector space spanned by all commutators in $\mathcal{M}$. If $a - b \in K$, we write $a \sim b$.

An element ρ of $\mathcal{M}$ is called a *trace integral* of (2.5), if $D_t(\rho) \sim 0$. Trace integrals ρ_1 and ρ_2 are called equivalent if $\rho_1 - \rho_2 \sim 0$. By definition, the degree of a trace integral is the minimal degree of elements from the corresponding equivalent class.

Low degree trace integrals of (1.3) can be found straightforwardly. In particular, there are no integrals of degrees 1 and 3, h is the only integral of degree 2, three linearly independent trace integrals of degree 4 are given by h^2, I and I^{-1}. Detailed information of higher degree integrals can be found in Section 4.

To generate infinite sequences of trace integrals for (1.3) the following two procedures can be applied.

Procedure 1. Suppose two Laurent polynomials $H, A \in \mathcal{M}$ satisfy

$$D_t(H) = [A,\, H] \tag{2.9}$$

where D_t is the derivation corresponding to (1.3). The involution (2.7) generates the pair $\overline{A}, \overline{H}$ that is also satisfying (2.9). It follows from (2.9) that H is a trace integral for (1.3). Since H^k also satisfy (2.9) for any natural k the elements H^k are trace integrals as well. In the matrix case this is equivalent to the fact that the spectrum of H is preserved under the flow (1.3). Thus for any pair (H_i, A_i) satisfying (2.9) the elements

$$\rho_{ik} = H_i^k, \qquad \text{and} \qquad \overline{\rho_{ik}} = \overline{H_i}^k \tag{2.10}$$

are trace integrals for any natural k.

It is easy to see that if H, A satisfy (2.9), then for any invertible element $g \in \mathcal{M}$ the conjugation

$$H \to gHg^{-1}, \qquad A \to gAg^{-1} + g_t g^{-1} \tag{2.11}$$

leads to another pair satisfying (2.9). It is clear that conjugated pairs produce the same trace integrals (2.10).

Apart from h given by (1.2) satisfying the relation (2.9) with $A = -u^{-1} - v$ we found several more low degree pairs (H_i, A_i) satisfying (2.9).

The corresponding elements $H_i, \quad i=1,..,11$ are given by $H_i = h + a_i$, where

$$\begin{array}{lllllll}
a_1 &=& [u^{-1}, vu] &=& v\ (S^2 I^{-1} - 1), & a_7 &=& SI + a_2 + a_4, \\
a_2 &=& [v\ \ , u^{-1}v^{-1}] &=& u^{-1}(\ \ I\ \ \ - 1), & a_8 &=& SI + a_2 + [v, u^{-1}v^2], \\
a_3 &=& [v\ \ , uv^{-1}] &=& u\ (S^3 I^{-1} - 1), & a_9 &=& SI + a_4 + [u^{-1}, u^{-1}v^{-1}u], \\
a_4 &=& [u^{-1}, v^{-1}u] &=& v^{-1}(S\ \ I\ \ \ - 1), & a_{10} &=& S^2 I + [v^{-1}, uv], \\
a_5 &=& a_1 + a_4, & & & a_{11} &=& I + [u, vu^{-1}] \\
a_6 &=& a_2 + a_3, & & & & &
\end{array}$$

Here S stands for a cyclic shift of factors in a monomial, i.e. $S(abc...z) = bc...za$, and I is given by (2.8). Sets of $H_i, \overline{H_i}$ that are conjugate to each other are $\{h, \overline{H_5}, \overline{H_6}\}$, $\{H_1, H_3, \overline{H_2}, \overline{H_4}\}$, $\{H_7\}$, $\{H_8, H_9, H_{10}, H_{11}\}$ and the involuted versions of these 4 groups. In addition to h, A three of the pairs H_i, A_i representing these groups up to involution and conjugation are:

$$\begin{array}{ll}
H_1 = u + u^{-1} + v^{-1} + u^{-1}v^{-1} + u^{-1}vu, & A_1 = u - v + v^{-1} + u^{-1}v^{-1}, \\
H_7 = u + v + u^{-1}v^{-1}u + vu^{-1}v^{-1} + u^{-1}v^{-1} + vu^{-1}v^{-1}u, & A_7 = v^{-1} - v, \\
H_{11} = u + u^{-1} + v^{-1} + u^{-1}v^{-1} + uvu^{-1} + uvu^{-1}v^{-1}, & A_{11} = -u^{-1}.
\end{array}$$

Procedure 2. Suppose we have two Laurent polynomials P and Q such that

$$D_t(P) = [I,\, Q], \tag{2.12}$$

where I is given by (2.8). Then it is easy to see that the elements PI^k are trace integrals of (1.3) for any integer k. If $P = [I, R]$ for some R then these trace integrals are equivalent to zero and thus trivial. An example for P that yields non-trivial integrals is given by

$$P = \underline{u}uvu^{-1}v^{-1} + u\underline{v}vu^{-1}v^{-1} + uv\underline{u^{-1}}u^{-1}v^{-1} + uvu^{-1}\underline{v^{-1}}v^{-1} + \underline{u^{-1}v^{-1}}uvu^{-1}v^{-1}. \tag{2.13}$$

Some factors are underlined only to illustrate how P can be looked at as a synthesis of the terms of h from (1.2) (underlined) with I (not underlined) which is referred to in section 4.

Notice, that (2.12) looks similar to local conservation laws of the form $\rho_t = \sigma_x$ for the evolutionary equations of the Korteweg-de Vries type. On the right hand side of (2.12) we see the operator ad_I defined by $\mathrm{ad}_I(w) = [I, w]$. It is well known that this operator is a derivation (i.e. a linear operator that satisfies the product rule and commutes with ∂_t).

3 The pre-Hamiltonian operator

In the previous section we found several infinite sequences of trace integrals for (1.3). Here we construct the sequences of corresponding symmetries via a so called pre-Hamiltonian operator. This operator is defined in terms of left and right multiplication operators.

For any Laurent polynomial $a \in \mathcal{M}$ we denote by L_a and R_a the operators of left and right multiplications on $\mathcal{M}$:

$$L_a(x) = a\,x, \qquad R_a(x) = x\,a\,.$$

It is clear that $L_{ab} = L_a L_b$ and $R_{ab} = R_b R_a$. It follows from the associativity of the algebra $\mathcal{M}$ that $R_a L_b = L_b R_a$.

The algebra of all left and right multiplication operators is generated by $L_{x_i}, L_{x_i^{-1}}, R_{x_i}, R_{x_i^{-1}}$. We denote this associative algebra by $\mathcal{O}$ and call it the *algebra of local operators*.

For any element $a = a(\mathbf{x}) \in \mathcal{M}$ we define an $1 \times N$-matrix $\mathbf{a}_*$ with entries being elements of $\mathcal{O}$ by the following identity:

$$\frac{d}{d\epsilon} a(\mathbf{x} + \epsilon\, \delta\mathbf{x})|_{\epsilon=0} = \mathbf{a}_*(\delta\mathbf{x})\,. \tag{3.14}$$

For example, for h from (1.2) we have

$$\mathbf{h}_* = (\ 1 - L_{u^{-1}} R_{u^{-1}} - L_{u^{-1}} R_{u^{-1}} R_{v^{-1}}\ ,\ \ 1 - L_{v^{-1}} R_{v^{-1}} - L_{u^{-1}} L_{v^{-1}} R_{v^{-1}}\ ).$$

It is easy to see that

$$D_t(a) = \mathbf{a}_*(\mathbf{F}), \tag{3.15}$$

where D_t is the derivation associated with (2.5) and $\mathbf{F} = (F_1, ..., F_N)^T$ is the right hand side of (2.5).

For any vector $\mathbf{a} = (a_1, ..., a_N)^T$, $a_i \in \mathcal{M}$ we define the Fréchet derivative operator $\mathbf{a}_*$ as the $N \times N$-matrix with rows $(\mathbf{a_1})_*, ..., (\mathbf{a_N})_*$.

For any two vectors $\mathbf{p} = (p_1, ..., p_N)^T$, $\mathbf{q} = (q_1, ..., q_N)^T$, $p_i, q_i \in \mathcal{M}$ we put

$$\langle \mathbf{p}, \mathbf{q} \rangle = p_1 q_1 + \cdots + p_N q_N.$$

Let $a(\mathbf{x}) \in \mathcal{M}$. Then $\mathbf{grad}\,(a)$ is the vector uniquely defined by:

$$\frac{d}{d\epsilon} a(\mathbf{x} + \epsilon\, \delta\mathbf{x})|_{\epsilon=0} \sim \langle \delta\mathbf{x}, \mathbf{grad}\,(a(\mathbf{x})) \rangle\,.$$

We will denote by $grad_{x_1}(a), ..., grad_{x_N}(a)$ the components of the vector $\mathbf{grad}$ (a). It is easy to see that if $a \sim b$, then $\mathbf{grad}\,(a) = \mathbf{grad}\,(b)$. This means that $\mathbf{grad}\,(a)$ is well-defined for trace integrals.

For example, for the function h given by (1.2) we have

$$\mathbf{grad}\, h = (1 - u^{-2} - u^{-1} v^{-1} u^{-1},\quad 1 - v^{-2} - v^{-1} u^{-1} v^{-1})^T.$$

It follows from the definition of an infinitesimal symmetry (2.6) with symmetry generator $\mathbf{G} = (G_1, .., G_N)^T$ for a system (2.5) that $D_t D_\tau \mathbf{x} = D_\tau D_t \mathbf{x}$, i.e.

$$D_t \mathbf{G} = D_\tau \mathbf{F} = \mathbf{F}_*(\mathbf{G}) \tag{3.16}$$

(from (3.15)) is a linearized equation satisfied by $\mathbf{G}$, where $\mathbf{F}_*$ is the Fréchet derivative of the right hand side of (2.5) (cf. [6]).

An $N \times N$ matrix $\mathcal{P}$ with entries from $\mathcal{O}$ is called a *pre-Hamiltonian operator* for equation (2.5) if

$$D_t(\mathcal{P}) = \mathbf{F}_* \mathcal{P} + \mathcal{P} \mathbf{F}_*^\star. \tag{3.17}$$

Here the adjoint operation $\star$ on $\mathrm{Mat}_N(\mathcal{O})$ is uniquely defined by the formula

$$\langle \mathbf{p}, \mathbf{Q}(\mathbf{q}) \rangle \ \sim \ \langle \mathbf{Q}^\star(\mathbf{p}), \mathbf{q} \rangle, \tag{3.18}$$

where $\mathbf{Q} \in \mathrm{Mat}_N(\mathcal{O})$, $p_i, q_i \in \mathcal{M}$.

Relation (3.17) can be rewritten in the form

$$(D_t - \mathbf{F}_*)\, \mathcal{P} = \mathcal{P}\, (D_t + \mathbf{F}_*^\star)\,. \tag{3.19}$$

It can be shown (cf. [6]) that for any trace integral a of (2.5) the vector $\mathbf{b} = \mathbf{grad}(a)$ satisfies the equation $D_t(\mathbf{b}) + \mathbf{F}_*^\star(\mathbf{b}) = 0$. Applying both sides of (3.19) to $\mathbf{b}$, we get that any pre-Hamiltonian operator maps gradients of integrals for (2.5) to symmetries.

Proposition. The following operator (cf. [3])

$$\mathcal{P} = \begin{pmatrix} R_u R_u - L_u L_u & L_u L_v + L_u R_v - L_v R_u + R_u R_v \\ L_u R_v - L_v L_u - L_v R_u - R_v R_u & L_v L_v - R_v R_v \end{pmatrix} \tag{3.20}$$

is a pre-Hamiltonian operator for equation (1.3). The proof consists of the straightforward verification of relation (3.17) using

$$\mathbf{F}_* = \begin{pmatrix} R_v - R_{v^{-1}} & L_u + L_u L_{v^{-1}} R_{v^{-1}} + L_{v^{-1}} R_{v^{-1}} \\ -L_v - L_v L_{u^{-1}} R_{u^{-1}} - L_{u^{-1}} R_{u^{-1}} & -R_u + R_{u^{-1}} \end{pmatrix}$$

$$\mathbf{F}_*^\star = \begin{pmatrix} L_v - L_{v^{-1}} & -R_v - L_{u^{-1}} R_{u^{-1}} R_v - L_{u^{-1}} R_{u^{-1}} \\ R_u + L_{v^{-1}} R_{v^{-1}} R_u + L_{v^{-1}} R_{v^{-1}} & -L_u + L_{u^{-1}} \end{pmatrix}$$

computed from (3.14), (3.18).

Applying operator (3.20) to $\mathbf{grad}\, h$, we get (up to a factor of 2) the right hand side of system (1.3). In this sense $h/2$ plays the role of a Hamiltonian for (1.3). However, the bracket

$$\{a, b\} =< \mathbf{grad}\, a,\, \mathcal{P}\, \mathbf{grad}\, b >$$

defined on traces of Laurent polynomials does not satisfy the Jacobi identity for $\mathcal{P}$ from (3.20) and a true Hamiltonian structure for (1.3) is yet unknown.

4 The Lax pair

Consider

$$\mathbf{L} = \begin{pmatrix} v^{-1} + u & v\lambda + (v^{-1}u^{-1} + u^{-1} + 1) \\ v^{-1} + u\frac{1}{\lambda} & v^{-1}u^{-1} + u^{-1} + v + \frac{1}{\lambda} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} v^{-1} - v + u & \lambda v \\ v^{-1} & u \end{pmatrix}, \tag{4.21}$$

where λ is a (scalar) spectral parameter. Then the relation

$$D_t\mathbf{L} = [\mathbf{A}, \mathbf{L}]$$

is equivalent to the Kontsevich system (1.3). Although the pairs H_i, A_i described in Section 2 do also satisfy an equation similar to (2.9), they are not Lax pairs for (1.3) since in this case (2.9) follows from (1.3) but not vice versa.

The Lax pair can be replaced by any equivalent one obtained through a conjugation (2.11), where g is an arbitrary invertible Laurent 2×2-matrix. Other equivalence transformations are $\mathbf{L} \to P_1(\mathbf{L}),\ \mathbf{A} \to \mathbf{A} + P_2(\mathbf{L})$, where P_i are polynomials with constant λ-dependent coefficients, and arbitrary transformation $\lambda \to f(\lambda)$.

Applying the involution (2.7) to the Lax pair (4.21), we get a dual one:

$$\overline{\mathbf{L}} = \begin{pmatrix} u^{-1}+v & u\lambda + (u^{-1}v^{-1}+v^{-1}+1) \\ u^{-1}+v\frac{1}{\lambda} & u^{-1}v^{-1}+v^{-1}+u+\frac{1}{\lambda} \end{pmatrix}, \qquad \overline{\mathbf{A}} = \begin{pmatrix} u^{-1}-u+v & \lambda u \\ u^{-1} & v \end{pmatrix}. \tag{4.22}$$

It is possible to verify that the dual Lax pair is not equivalent to (4.21) but each one of $\mathbf{L}$ and $\overline{\mathbf{L}}$ generates the same vector space of first integrals, verified up to degree 14.

As usual, the traces $tr\ \mathbf{L}^m = (\mathbf{L}^m)_{11} + (\mathbf{L}^m)_{22}$ generate trace integrals of motion. In particular, $tr\ \mathbf{L}$ yields $v^{-1}+u+v^{-1}u^{-1}+u^{-1}+v$, which is equivalent to h from (1.2). In contrast to (2.9) each power of $\mathbf{L}$ gives us several trace integrals since $tr\ \mathbf{L}^m$ is a polynomial in λ, λ^{-1} with all coefficients being trace integrals. We verified that all trace integrals of degree ≤ 12 for (1.3) are generated in such a way.

Table 1 shows all integrals of degree d generated from $tr\ \mathbf{L}^m,\ \ m \leq 14$ that are not generated from $tr\ \mathbf{L}^i,\ i < m$. The following statements assume that all trace integrals have been reduced modulo lower degree trace integrals.

Each integral is represented by a $\star$, $\circ$ or $\bullet$ and is located in one diagonal of table 1. Each diagonal starts in a table entry which shows a number k indicating that the integral is I^k. For the single $\star$ - diagonal is $k=0$. $\circ$ - diagonals have $k<0$ and start in row $3|k|$, i.e. I^k result from $tr\ \mathbf{L}^{3|k|}$ and $\bullet$ - diagonals have $k>0$ and start in row $4k$, i.e. I^k result from $tr\ \mathbf{L}^{4k}$.

The second integral in a $\bullet$ - diagonal is PI^{k-1} with P from (2.13) being a special type of product of I and h (see footnote to (2.13)). The next integral in each $\bullet$ - diagonal is obtained from multiplying the terms of h^2 in a similar fashion to I^k. More generally, it appears that $\bullet$ - integrals of degree d in a diagonal starting with $I^k,\ k>0$ are composed out of I^k and $h^{(d-4k)/2}$ and $\circ$ - integrals of degree d in a diagonal starting with $I^k,\ k<0$ are composed out of I^k and $\overline{h}^{(d+4k)/2}$.

The different powers of λ in $tr\ \mathbf{L}^m$ have the following first integrals as coefficients. $\star$ - integrals are the coefficients of λ^0. $\circ$ - integrals of degree d resulting from $\mathbf{L}^m$ are the coefficients of $\lambda^{(d-2m)/2}$. $\bullet$ - integrals of degree d resulting from $\mathbf{L}^m$ are the coefficients of $\lambda^{(2m-d)/4}$.

All $\star$ - integrals are invariant under involution (2.7). All other first integrals come in pairs, one $\circ$ - and one $\bullet$ - integral, both are involution symmetric to each other and therefore in the same column.

Using $\overline{\mathbf{L}}$ instead of $\mathbf{L}$ does generate a table with $\circ$ and $\bullet$ - entries interchanged. Replacing $\lambda \to f(\lambda)$ in L (4.21) does not change the table.

Applying the pre-Hamiltonian operator (3.20) to $\star$ - integrals gives involution invariant symmetries of the same degree. Applying (3.20) to an involution symmetric pair of first integrals gives 2 symmetries that are one degree higher than the first integral and that are also involution symmetric to each other.

Traces of H_j^k (2.10) have been verified to be linear combinations of integrals of table 1. For example,

- first integrals $tr\ h^{2m+k},\ n = 0, 1, .., \ k = 0, 1$ are of degree $2(2m + k)$ and require only up to $\mathbf{L}^{2m+k}$ to be derived,
- first integrals $tr(h+a_1)^{2m+k},\ m = 0, 1, .., \ k = 0, 1$ are also of degree $2(2m+k)$ (although $h + a_1$ is of degree 3) but require up to $\mathbf{L}^{4m+k}$ to be derived,
- first integrals $tr(h + a_7)^m$ and $tr(h + a_{11})^m,\ m = 0, 1, ..$ are of degree $4m$ and require up to $\mathbf{L}^{3m}$ to be derived.

All first integrals generated by the Lax pair operator $\overline{\mathbf{L}}(= \mathbf{L}|_{u\leftrightarrow v})$ are linear combinations of first integrals generated from $\mathbf{L}$ and vice versa.

If a given first integral F of degree $2(2m + k),\ m = 0, 1, ..., \ k = 0, 1$ is to be expressed as a linear combination of first integrals computed from $\mathbf{L}$ and $\overline{\mathbf{L}}$ then it requires at least $\mathbf{L}^{2m+k}$ or $\overline{\mathbf{L}}^{2m+k}$ and at most $\mathbf{L}^{3m+k}$ and $\overline{\mathbf{L}}^{3m+k}$. If F is to be expressed as a linear combination of first integrals computed from $\mathbf{L}$ alone then it requires at most $\mathbf{L}^{4m+k}$.

The table of first integrals with its straightforward extension appears to be complete because all symmetries of degree up to 16 have been verified to be generated from integrals of this table and the pre-Hamiltonian operator (3.20).

Acknowledgements. Authors thank M. Kontsevich and V. Sokolov for fruitful discussions. E. Schrüfer is thanked for discussions and computer algebra support. O.E. is grateful to IHES for hospitality. She was partially supported by the RFBI grant 11-01-00341-a.

$m\backslash d$	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28
0	$\star^0$														
1		$\star$													
2			$\star$												
3			$\circ^{-1}$	$\star$											
4			$\bullet^{+1}$	$\circ$	$\star$										
5				$\bullet$	$\circ$	$\star$									
6					$\bullet\circ^{-2}$	$\circ$	$\star$								
7						$\bullet\circ$	$\circ$	$\star$							
8					$\bullet^{+2}$		$\bullet\circ$	$\circ$	$\star$						
9						$\bullet$	$\circ^{-3}$	$\bullet\circ$	$\circ$	$\star$					
10							$\bullet$	$\circ$	$\bullet\circ$	$\circ$	$\star$				
11								$\bullet$	$\circ$	$\bullet\circ$	$\circ$	$\star$			
12							$\bullet^{+3}$		$\bullet\circ^{-4}$	$\circ$	$\bullet\circ$	$\circ$	$\star$		
13								$\bullet$		$\bullet\circ$	$\circ$	$\bullet\circ$	$\circ$	$\star$	
14									$\bullet$		$\bullet\circ$	$\circ$	$\bullet\circ$	$\circ$	$\star$

Table 1: # of new trace first integrals of degree d generated from $tr\ \mathbf{L}^m$ not generated from $tr\ \mathbf{L}^i,\ i < m$.

References

[1] Kontsevich, M., private communication.

[2] Veselov, A.P., What is an integrable mapping, *in the book "What is Integrability?", ed. E.V. Zakharov, Springer–Verlag, New York*, 250–272, 1991.

[3] Mikhailov, A.V. and Sokolov, V.V., Integrable ODEs on Associative Algebras, *Comm in Math Phys.*, **211**, 231-251, 2000.

[4] Efimovskaya, O.V., Integrable cubic ODEs on Associative Algebras, *Fundamentalnaya i Prikladnaya Matematikaka*, **8**, no. **3**, 705-720, 2002.

[5] Olver, P.J. and Sokolov, V.V., Integrable evolution equations on associative algebras, *Comm. in Math. Phys.*, 1998, **193**, no.**2**, 245-268.

[6] Olver, P.J., Applications of Lie Groups to Differential Equations, Second Edition, Graduate Texts in Mathematics, **107**, *Springer–Verlag, New York*, 1993.