4.3.1 Investigating analytic properties of DEs

General description To investigate non-linear differential equations (DEs) or partial differential equations (PDEs), for which no general solution techniques are known, one either relies on numerics or, if one is interested in exact results, one investigates special properties, like symmetries or conservation laws. Any such results may give a better understanding of the problem, provide the basis for using more adequate numerical algorithms or even solve the problem analytically.

The circumstance that special situations are investigated (symmetries,...), which may but need not be present, has the consequence that the conditions for these properties lead to an overdetermined system of equations, usually PDEs.

To solve overdetermined PDE systems, the package CRACK contains about a dozen modules for the

- integration of different types of PDEs,
- direct and indirect separation of PDEs,
- computation of a Pseudo Differential Gröbner Basis,
- substitution of equations into each other,
- length reduction of equations,
- solution of underdetermined linear ODEs,
- reduction of redundant arbitrary constants and functions in solutions of DEs.

They can work together automatically or be used interactively driven by a menu. The programs LIEPDE, APPLYSYM and CONLAW use the package CRACK to solve the PDE-systems they generate.

CONLAW is a package of four programs implementing four different approaches for the computation of first integrals of single or systems of ordinary DEs (ODEs) or conservation laws for single or systems of PDEs [7], [5].

LIEPDE is a program for the determination of infinitesimal point-, contact- and generalized higher order symmetries of DEs and DE-systems.

With the program APPLYSYM point symmetries, computed with LIEPDE, can be integrated to yield symmetry- and similarity variables. After a transformation of the DEs into these variables, symmetry variables themselves disappear which allows the lowering of the order of ODEs or the reduction of the number of variables in the search of special solutions to PDEs.

System requirements The required system is REDUCE, version 3.6., strictly speaking the PSL version of Reduce as distributed by the Konrad Zuse Institut / Berlin. Work on compatibility issues aims to provide a CSL Reduce compatible version in near future.

The `*.log` files (see below) are produced from test files `*.tst` in a 8MB session running REDUCE, version 3.6 under LINUX.

Availability For each of the four packages CRACK, LIEPDE, APPLYSYM and CONLAW there are test files `*.tst`, log files `*.rlg`, manual descriptions `*.tex` and source files `*.red` freely available from `ftp.maths.qmw.ac.uk`. Currently (Sep. 1998) two web demos allow the use of CRACK and CONLAW for problems of restricted size, accessible from `http://cathode.maths.qmw.ac.uk/demos.html`. A publication list related to these programs and algorithms can be found at `http://www.maths.qmw.ac.uk/~tw`.

Target topics CRACK and related application programs are applicable whenever smooth analytic properties of differential equations or geometric objects including manifolds are investigated. With "smooth" we mean properties that can be described by differential equations which are valid in an open neighbourhood in contrast to problems from Discrete Mathematics like "Which integers satisfy ...". In this sense CONLAW, LIEPDE and APPLYSYM are just examples for applications of CRACK.
Comparisons with other systems The strengh of the package CRACK in comparison with other packages also aiming at the solution of overdetermined PDE systems is based on

- a wide range of algorithms and modules mentioned above,
- the possibility to run batch jobs as well as to work interactively,
- a high flexibility in the way the above mentioned modules interact.

What distinguishes the programs in CONLAW from other programs is that they allow the computation of

- conservation laws (CLs) for ODEs as well as PDEs of a non-restricted number of independent variables,
- CLs which are non-polynomial, even non-rational in the integrating factors and conserved quantities,
- CLs which have an explicit dependence on the independent variables,
- CLs for which an ansatz for the integrating factor and/or the first integral/conserved current can be made.

Examples of computations that become possible with CONLAW including new conservation laws are given in [7], [5]. A characterization of other programs for the computation of conservation laws is given in [1].

A comprehensive collection of references related to the computation of symmetries is given in [2]. The high efficiency of LIEPDE in comparison with other programs comes from

- using CRACK for solving the symmetry conditions,
- the ability to formulate the overdetermined symmetry conditions in portions and to solve them successively before the next conditions are formulated which are reduced in size and easier to solve as a consequence.

User base CRACK and related programs are available for about a decade in the Reduce network library and have been distributed as part of Reduce in recent years.
Applications The programs CONLAW, LIEPDE, APPLYSYM are applicable to investigate any kind of differential equation or system of differential equations (apart from complexity issues of course, when PDEs with many functions of many independent variables are to be investigated wrt. symmetries or conservation laws of high order). Examples are given in the `*.tst` files and in [7], [5].

These programs are themselves applications of CRACK and many other applications of CRACK can be thought of, for example, 1. the inverse variational problem, i.e. the question whether for a given system of DEs there is an equivalent variational principle, 2. the consistency of additional constraints added to a set of equations.

CRACK has been used in General Relativity either to investigate a special ansatz to find exact solutions [4], [3] of the field equations or to characterise space times by determining their symmetries [8], [6].

LIEPDE has been used in the study of PDEs playing a role in the description of weather processes.

Acknowledgement Andreas Brand is the author of a number of core modules of CRACK. The currently used data structure and program structure of the kernel of CRACK are due to him.

Francis Wright contributed a module that provides simplifications of expressions involving symbolic derivatives and integrals. Also, CRACK makes extensive use of the REDUCE program ODESOLVE written by Malcolm MacCallum and Francis Wright.

Arrigo Triulzi provided a module to support the use of different total orderings of derivatives in doing pseudo differential Gröbner basis computations.

Work on this package has been supported by the Konrad Zuse Institute/Berlin through a fellowship of T.W.. Winfried Neun and Herbert Melenk are thanked for many discussions and constant support.

Anthony Hearn provided free copies of REDUCE to us as a REDUCE developers group which also is thankfully acknowledged.

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