The computation of conservation laws for geodesic motion in curved spaces and for arbitrary PDE systems

Thomas Wolf
Queen Mary & Westfield College, University of London,
Mile End Road, London E1 4NS, UK
email: T.Wolf@maths.qmw.ac.uk

August 31, 1999

This report describes current work on problems related to the computation of conservation laws. Although the computation of conservation laws of geodesic motion in curved space and the computation of conservation laws of arbitrary systems of partial differential equations (PDEs) are rather different, in both cases overdetermined PDE-systems have to be solved which is accomplished with the same computer algebra package CRACK (included on the books CD).

The computation of Killing vectors and Killing tensors

If u^i is the velocity of a test particle moving in a curved space then "force free" (geodesic) motion is described by $u^i_{;j}u^j = 0$ where ';' denotes covariant differentiation and summation over equal indices is performed. Any constants of this motion that are polynomial in u^i and that are independent of the path parameter have the form

$$\text{const} = K_{i_1 \ldots i_r} u^{i_1} \ldots u^{i_r}.$$

The conditions on the tensor $K_{i_1 \ldots i_r}$ (called a Killing vector if the rank is one, otherwise Killing tensor) which is completely symmetric in its r indices are

$$K_{(i_1 \ldots i_r ; i_{(r+1)})} = 0 \tag{1}$$

where () stands for symmetrization.

The need to compute Killing tensors is not only restricted to problems in General Relativity or Differential Geometry. For any dynamical system described by a Hamiltonian

$$H = \frac{1}{2} g^{ij}(x^k) p_i p_j + V(x^k)$$

with Jacobi metric g^{ij}, momentum p_i and potential $V(x^k)$ the question of the existence of first integrals

$$F = K_s(p)^s + K_{s-1}(p)^{(s-1)} + \ldots + K_0, \qquad \text{(indices suppressed)}$$

polynomial in the momentum p with completely symmetric coefficient tensors of rank $s, s-1, \ldots$ requires the solution of the Killing tensor condition for K_s and K_{s-1}. The other K_{s-i} are calculated from them afterwards.

Another application is the question of separability of the Hamilton-Jacobi equation

$$g^{ij}\partial_{x^i}W\partial_{x^j}W = E, \quad g^{ij} = g^{ji}, \ \ 1 \le i, j \le n$$

for an n-dimensional Hamiltonian system, which leads to the question of the existence of enough Killing tensors in a space with metric g^{ij}.

One way to solve Killing tensor equations (1) is to use the program `CLASSYM` [3] to formulate the equations for a given metric including additional integrability conditions and to use the package `CRACK` to solve them all. Such overdetermined PDE-systems can easily have thousands of terms.

To give an example, recent improvements in lowering the memory consumption during the formulation and application of integrability conditions within `CRACK` (computation of the differential Gröbner basis) have made it possible to compute the 4 Killing vectors (KV) and 13 Killing tensors (KT) of rank 2 of the Kimura metric [2]

$$g_{tt} = -r^2/b, \ \ g_{rr} = 1/(rb)^2, \ \ g_{\theta\theta} = r^2, \ \ g_{\phi\phi} = r^2\sin^2(\theta),$$

where all other $g_{ij} = 0$ and $i, j = t, r, \theta, \phi$. The 4 KV come from its static spherical symmetry and among the 13 KT are 10 reducible KT which are all symmetrized products of 2 of the 4 KV, one trivial KT which is the metric g_{ij} but also 2 non-trivial KT:

$$K_{tt} = 2tr^4/b, \ K_{tr} = r/b^2, \ K_{rr} = -2t/(rb)^4, \ K_{\phi\phi} = -2tr^4\sin^2(\theta) \quad \text{and}$$

$$K_{tt} = t^2r^4/b + r^2/b^2, \ K_{tr} = tr/b^2, \ K_{\theta\theta} = -t^2r^4, \ K_{\phi\phi} = -t^2r^4\sin^2(\theta).$$

In a different approach an algorithm was developed [5] to write conditions (1) for KT of rank r and the integrability conditions of order up to r as a system of structural equations

$$(F_A)_{;k} = \sum_B \Gamma_{kAB}F_B \tag{2}$$

for unknowns F_A which are the components of a Killing tensor $K_{i_1\ldots i_r}$ and its symmetrized covariant derivatives of order up to r. The Γ_{kAB} are expressions in the Riemann tensor for the given metric and its first r derivatives. One advantage of this formulation is that all integrability conditions are linear algebraic conditions for the F_A. After completing

- lengthy tensor algebra computations with an especially written computer tensor program, to formulate the large tensorial system (2),
- programs that compute the relevant tensors in (2) for a given metric and in this way formulate (2) for a given metric,
- a module to compute integrability conditions for systems of the structure of (2),

the current efforts concentrate on computer algebra algorithms to solve large linear algebraic systems (the integrability conditions) which are too big for currently available algorithms and software.

The computation of conservation laws of ODEs and PDEs

For a given system of differential equations $\Delta_\mu = 0$ for unknown functions $u = (u^1, u^2, \ldots, u^\nu)$ of independent variables $x = (x^1, x^2, \ldots, x^p)$ conservation laws can be formulated as a condition

$$\mathrm{Div}\, P = 0 \qquad (\mathrm{mod}\ \Delta = 0,\ D_i\Delta = 0, \ldots) \tag{3}$$

for a conserved current $P = (P^1, \ldots, P^p)$. Another possible formulation is the condition

$$\mathrm{Div}\, P = \sum_\nu Q^\nu \Delta_\nu \tag{4}$$

for a conserved current P and characteristic functions $Q = (Q^1, \ldots, Q^\mu)$. Eliminating $\mathrm{Div}P$, and hence P with the Euler-Lagrange operator we obtain conditions for the Q_ν

$$\forall \nu: \qquad 0 = E_\nu\left(\sum_\mu Q_\mu \Delta_\mu\right) = \sum_J (-D)_J \left(\frac{\partial}{\partial u^\nu_J} \sum_\mu Q_\mu \Delta_\mu\right) \tag{5}$$

which on the space of solutions of $\Delta_\mu = 0$ give the necessary conditions

$$0 = \sum_{\mu,J} (-D)_J \left(Q_\mu \frac{\partial \Delta_\mu}{\partial u^\nu_J}\right)\Bigg|_{\Delta_\mu = 0} \qquad \forall \nu. \tag{6}$$

Equations (6) are the conditions for adjoint symmetries of $\Delta = 0$.

Each of the approaches (3)-(6) has different advantages, therefore four programs ConLaw1/2/3/4 have been written to be able to use the most appropriate approach for any given problem (see [6],[7]). Compared with other programs, e.g. [1], ConLaw can be used to find conservation laws that are non-polynomial (next example) and have an explicit variable dependence. The number of equations and variables are only restricted by complexity issues.

Examples: The Liouville equation for a function $u = u(x,y)$ reads $\Delta = u_{xy} - e^u$. Conservation laws with Q of order zero found by ConLaw3 are

$$(f_x + f u_x)\Delta = D_x(-e^u f) + D_y(f_x u_x + f u_x^2/2), \quad f = f(x) \quad \text{arbitrary}$$

with equivalent laws for $x \leftrightarrow y$.

To give another example, with ConLaw2 it was possible for the Kadomtsev - Petviashvili equation in its potential form

$$0 = [v_t + v_{3x} + v_x^2]_x - v_{2y}, \quad v = v(t,x,y)$$

to find non-trivial conservation laws, like one given through

$$Q = -c_{4t} y^4 - 12 c_{3t} x y^2 + 24 c_{2t} y^2 v_x - 12 c_{2t} x^2 + 48 c_t x v_x + 96 c_t y v_y + 48 c_t v + 144 c v_t$$

with $c = c(t)$ being an arbitrary function. (The conserved current P has been omitted due to its length.)

Allowing the computation of non-polynomial conservation laws, even involving arbitrary functions enabled the linearization of non-linear evolutionary PDE-systems of the Schrödinger type as described in [8].

Demonstration versions of the programs Crack and ConLaw are accessible online under `http://cathode.maths.qmw.ac.uk` and the code including manuals on ftp.maths.qmw.ac.uk in the directory `pub/tw`.

References

[1] Göktaş, Ü. and Hereman, W. (1996) Symbolic Computation of Conserved Densities for Systems of Nonlinear Evolution Equations, Journal of Symbolic Computation, **24** (1997), 591-621.

[2] Kimura, M.,Tensor N.S. Vol. 30 (1976), 27-43.

[3] Wolf, T. and Grebot, G., Automatic symmetry investigation of space-time metrics, Int. J. of Mod. Phys. D, **3**, no. 1, 323, 1994.

[4] Wolf, T., The program CRACK for solving PDEs in General Relativity, in F.W. Hehl, R.A. Puntigam, and H. Ruder (Eds.): Relativity and Scientific Computing: Computer Algebra, Numerics, Visualization. Springer Verlag, 1996, p. 241-258.

[5] Wolf, T., Structural equations for Killing tensors of arbitrary rank, Comp. Phys. Comm., **115** (1998), 316 - 329.

[6] Wolf, T., Brand, A., Mohammadzadeh, M., Computer algebra algorithms and routines for the computation of conservation laws and fixing of gauge in differential expressions, J. Symb. Comp. 27 (1999), 221-238.

[7] Wolf, T., A comparison of four approaches to the calculation of conservation laws, preprint, 23 pages, (1998).

[8] Sokolov, V.V., Wolf, T., A symmetry test for quasilinear coupled systems, Inverse Problems, 15 (1999), L5-L11.