About Integrable Non-Abelian Laurent ODEs

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Outline

Non-commutative ODEs

First Integrals and Lax Pairs

Symmetries

Pre-Hamiltonian Operators

Recursion Operators

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Computer Runs

References
Examples of integrable matrix homogeneous ODEs

Integrable systems for unknown matrices $u(t)$ and $v(t)$ of arbitrary size are

$$u_t = uv, \quad v_t = -vu$$

(1)

and

$$u_t = v^2, \quad v_t = u^2.$$  

(2)
Examples of integrable matrix homogeneous ODEs

Integrable systems for unknown matrices \( u(t) \) and \( v(t) \) of arbitrary size are

\[
\begin{align*}
    u_t &= uv, \quad v_t = -vu \quad (1) \\
    u_t &= v^2, \quad v_t = u^2. \quad (2)
\end{align*}
\]

Another integrable system is

\[
\begin{align*}
    u_t &= u^2 \nu - \nu^2 u, \quad v_t = 0. \quad (3)
\end{align*}
\]

If \( u \) is a skew-symmetric and \( v \) is a diagonal matrix, then this system coincides with the \( n \)-dimensional Euler top. The integrability of this model was established by S.V. Manakov in 1976. The simplest first integrals are given by

\[
\begin{align*}
    l_{2,2} &= \text{Tr} \left( 2v^2 u^2 + vu v^2 u \right), \quad l_{2,3} = \text{Tr} \left( v^3 u^2 + v^2 u v^2 u \right),
\end{align*}
\]

where \( \text{Tr} \) stands for the trace of the matrix.
Block Matrix Equations

The cyclic reduction

\[ u = \begin{pmatrix} 0 & u_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & u_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & u_{n-1} \\ u_n & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & J_n \\ J_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & J_{n-1} & 0 \end{pmatrix} \]

where \( u_k \) and \( J_k \) are matrices converts (3) to the matrix Volterra equation

\[
\frac{d}{dt} u_k = u_k u_{k+1} J_{k+1} - J_{k-1} u_{k-1} u_k, \quad k \in \mathbb{Z}_k.
\]

If we assume \( n = 3, \ J_1 = J_2 = J_3 = I \) and \( u_3 = -u_1 - u_2 \) then the latter system is equivalent to (2) (Mikhailov, Sokolov).
Non-abelian Laurent Polynomials

In connection with the theory of non-commutative elliptic curves, M. Kontsevich considered the following discrete map

\[ u \rightarrow uvu^{-1}, \quad v \rightarrow u^{-1} + v^{-1}u^{-1}, \quad (4) \]

where \( u, v \) are non-commutative variables. His numerical computer experiments have shown that this map may be integrable and possess the Laurent property. The latter means that the right hand sides for all iterations of (4) are polynomials in \( u, v, u^{-1}, v^{-1} \).
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Kontsevich observed also that (4) is a discrete symmetry of the following non-abelian ODE system:

$$u_t = uv - uv^{-1} - v^{-1}, \quad v_t = -vu + vu^{-1} + u^{-1}$$

and conjectured that (5) is integrable itself.
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and conjectured that (5) is integrable itself.

Non-commutative polynomials in \( u, v, u^{-1}, v^{-1} \) like the right hand sides of (5) are called non-abelian Laurent polynomials.
Componentless Calculus

Consider two-component ODEs of the form

\[ u_t = P_1(u, v, u^{-1}, v^{-1}), \quad v_t = P_2(u, v, u^{-1}, v^{-1}) \] (6)

on the free associative algebra \( \mathcal{M} \) generated by two generators \( u \) and \( v \). Here \( P_i \) are some (non-commutative) polynomials from \( \mathcal{M} \).
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- $(u^{-1})_t$ and $(v^{-1})_t$ are given by

$$ (u^{-1})_t = -u^{-1}u_t u^{-1}, \quad (v^{-1})_t = -v^{-1}v_t v^{-1}. $$
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\( f_1, f_2 \in \mathcal{M} \) are called equivalent \( (f_1 \sim f_2) \) iff \( f_1 \) can be obtained from \( f_2 \) by cyclic permutations of factors in its monomials. The equivalence class of any \( f \) is denoted by \( \text{Tr} f \).
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- (The transpose of a polynomial of matrices is obtained by reversing the order of all factors in each monomial and replacing the factors by their transpose.)
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First Integrals

An element $I$ of $\mathcal{M}$ is called an $\mathcal{M}$-integral of (6), if $\frac{dI}{dt} = 0$.

Example: for the Kontsevich system (KS):

$I = uvu - v - 1$, $\frac{dI}{dt} = 0$.

An element $h$ of $\mathcal{M}$ is called a trace integral of (6), if $\frac{dh}{dt} \sim 0$. Trace integrals $h_1$ and $h_2$ are called equivalent if $h_1 - h_2 \sim 0$.

Example: $h = u + v + u - 1 + v - 1 + u - 1 v - 1$, $\text{Tr} \frac{dh}{dt} = 0$.

Example: In the associative algebra $\mathcal{Q}$ generated by $u$, $v$ with $uv = q vu$, where $q$ is a fixed constant (the "q-case") $J = u + qv + qu - 1 + v - 1 + u - 1 v - 1$ is a full integral for KS. ($J$ is a $q$-deformation of $h$.)

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First Integrals

An element $l$ of $\mathcal{M}$ is called an $\mathcal{M}$-integral of (6), if $\frac{dl}{dt} = 0$. Example: for the Kontsevich system (KS):

$$l = u v u^{-1} v^{-1}, \quad dl/dt = 0.$$  

An element $h$ of $\mathcal{M}$ is called a trace integral of (6), if $\frac{dh}{dt} \sim 0$. Trace integrals $h_1$ and $h_2$ are called equivalent if $h_1 - h_2 \sim 0$. Example: $h = u + v + u^{-1} + v^{-1} + u^{-1} v^{-1}$, $\text{Tr } dh/dt = 0$. 
First Integrals

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Example: In the associative algebra $\mathcal{Q}$ generated by $u, v$ with $uv = qv u$, where $q$ is a fixed constant (the “$q$-case”)

$$J = u + qv + qu^{-1} + v^{-1} + u^{-1}v^{-1}$$

is a full integral for KS. ($J$ is a $q$-deformation of $h$.)
Lax Pairs I

Two Laurent polynomials $L, A$ are called a Lax pair if they satisfy

$$L_t = [A, L]$$  \hspace{1cm} (7)

and if (7) is equivalent to the system (here KS).
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In that case also $L^k, A$ for any natural $k$ satisfy (7).
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and if (7) is equivalent to the system (here KS).

In that case also $L^k, A$ for any natural $k$ satisfy (7).

Taking the trace of $L_t^k = [A, L^k]$ gives

$$(\text{Tr} \ L^k)_t = \text{Tr} (L^k_t) = \text{Tr} [A, L^k] = \text{Tr} (AL^k) - \text{Tr} (L^k A) = 0$$

Each $L$ gives us an infinite series of trace integrals, one for each $k$ even if (7) is not equivalent to the system.
To find Lax pairs an ansatz for Laurent polynomials $L, A$ up to some degree has been made.
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The condition

$$L_t = [A, L]$$

leads to bi-linear polynomial (i.e. non-linear) algebraic systems for the indefinite coefficients in $L$ and $A$. 
Infinite Sequences of Trace Integrals I

We found 12 pairs of polynomials $L, A$ satisfying $L_t = [A, L]$ and thus providing 12 infinite sequences of trace first integrals. But $L, A$ were not equivalent to the Kontsevich system.
Infinite Sequences of Trace Integrals I

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Apart from $L = h (= u + v + u^{-1} + v^{-1} + u^{-1}v^{-1}), A = -v - u^{-1}$ the $L_i$ of the other pairs are given by $L_i = h + a_i$:

$$
\begin{align*}
    a_1 &= [u^{-1}, vu] \\
    a_2 &= [v, u^{-1}v^{-1}] \\
    a_3 &= [v, uv^{-1}] \\
    a_4 &= [u^{-1}, v^{-1}u] \\
    a_5 &= a_1 + a_4, \\
    a_6 &= a_2 + a_3, \\
    a_7 &= a_2 + a_4 + vu^{-1}v^{-1}u, \\
    a_8 &= a_2 + vu^{-1}v^{-1}u + [v, u^{-1}v^2], \\
    a_9 &= a_4 + vu^{-1}v^{-1}u + [u^{-1}, u^{-1}v^{-1}u], \\
    a_{10} &= u^{-1}v^{-1}uv + [v^{-1}, uv], \\
    a_{11} &= uvu^{-1}v^{-1} + [u, vu^{-1}]
\end{align*}
$$
Due to the discrete symmetry (inversion) of the Kontsevich system $u \leftrightarrow v, t \leftrightarrow -t$ this inversion applied to $L_i, A_i$ gives 12 new pairs $\bar{L}_i, \bar{A}_i$. For each pair $L_i, A_i$ and any invertable Laurent polynomial $f$ a new pair $\hat{L}_i, \hat{A}_i$ is defined by conjugation

$$\hat{L}_i = f L_i f^{-1} \quad \hat{A}_i = f A_i f^{-1} + f t f^{-1}$$

satisfying $\hat{L}_i t = [\hat{A}_i, \hat{L}_i]$. But, two $L_i, \hat{L}_i$ related by conjugation give the same trace integrals.

Identification through conjugation: 24 $L_i, A_i \rightarrow 8$ groups up to inversion $\rightarrow 4$ groups represented by $h, h + a_1, h + a_7, h + a_{11}$. 
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For each pair $L, A$ and any invertable Laurent polynomial $f$ a new pair $\hat{L}, \hat{A}$ is defined by conjugation

$$\hat{L} = fLf^{-1}$$
$$\hat{A} = fAf^{-1} + f_t f^{-1}$$

satisfying $\hat{L}_t = [\hat{A}, \hat{L}]$.
But, two $L, \hat{L}$ related by conjugation give the same trace integrals.
Infinite Sequences of Trace Integrals II

Due to the discrete symmetry (inversion) of the Kontsevich system $u \leftrightarrow v, t \leftrightarrow -t$ this inversion applied to $L_i, A_i$ gives 12 new pairs $\bar{L}_i, \bar{A}_i$.

For each pair $L, A$ and any invertable Laurent polynomial $f$ a new pair $\hat{L}, \hat{A}$ is defined by conjugation

$$\hat{L} = fL f^{-1}$$
$$\hat{A} = fAf^{-1} + ft f^{-1}$$

satisfying $\hat{L}_t = [\hat{A}, \hat{L}]$.

But, two $L, \hat{L}$ related by conjugation give the same trace integrals.

Identification through conjugation: $24 \ L_i, A_i \rightarrow 8$ groups up to inversion $\rightarrow 4$ groups represented by $h, h + a_1, h + a_7, h + a_{11}$. 
Another infinite sequence of first integrals:

If $P, Q \in \mathcal{M}$ satisfy

$$D_t(P) = [I, Q],$$

where $\frac{dl}{dt} = 0$ then not only $P$ is a trace integral but also $PI^k$ are trace integrals $\forall k$. 

A trivial $P$:

$$P = [I, R]$$

for some $R$ as $\text{Tr}(PI^k) = 0$.

A non-trivial $P$:

$$P = uvuv^{-1} - uvuv^{-1} + uvuv^{-1} - uvuv^{-1} + uvuv^{-1} - uvuv^{-1} + uvuv^{-1} - uvuv^{-1} + u^{-1}v^{-1}uvu^{-1}v^{-1}.$$ 

which is a merger of the full first integral $I = uvuv^{-1} - uvuv^{-1}$ and the trace first integral $h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1}$. 

Infinite Sequences of Trace Integrals III
Another infinite sequence of first integrals:
If $P, Q \in \mathcal{M}$ satisfy
\[ D_t(P) = [I, Q], \] (8)
where $dl/dt = 0$ then not only $P$ is a trace integral but also $PI^k$ are trace integrals $\forall k$.

trivial $P$: $P = [I, R]$ for some $R$ as $\text{Tr}(PI^k) = 0$. 

\[ \text{trivial } P: P = [I, R] \text{ for some } R \text{ as } \text{Tr}(PI^k) = 0. \]
Infinite Sequences of Trace Integrals III

Another infinite sequence of first integrals:
If \( P, Q \in \mathcal{M} \) satisfy
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D_t(P) = [I, Q],
\]
where \( dl/dt = 0 \) then not only \( P \) is a trace integral but also \( PI^k \) are trace integrals \( \forall k \).

trivial \( P \): \( P = [I, R] \) for some \( R \) as \( \text{Tr}(PI^k) = 0 \).

non-trivial \( P \):
\[
P = uuuvu^{-1}v^{-1} + uvvu^{-1}v^{-1} + uvu^{-1}u^{-1}v^{-1} + uvu^{-1}v^{-1}v^{-1} + u^{-1}v^{-1}uvv^{-1}v^{-1}.
\]
Another infinite sequence of first integrals:

If $P, Q \in \mathcal{M}$ satisfy

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where $dl/dt = 0$ then not only $P$ is a trace integral but also $PI^k$ are trace integrals $\forall k$.

trivial $P$: $P = [I, R]$ for some $R$ as Tr$(PI^k) = 0$.

non-trivial $P$:

$$P = uuvu^{-1}v^{-1} + uvvu^{-1}v^{-1} + uvu^{-1}u^{-1}v^{-1} + uvu^{-1}v^{-1}v^{-1}$$
$$+ u^{-1}v^{-1}uvu^{-1}v^{-1}. $$

which is a merger of the full first integral

$$I = uvu^{-1}v^{-1}$$

and the trace first integral

$$h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1}.$$
A proper Lax pair composed of $2 \times 2$ matrices $L, A$ was finally found:

\[
L = \begin{pmatrix}
  v^{-1} + u & v \lambda + (v^{-1} u^{-1} + u^{-1} + 1) \\
  v^{-1} + u \frac{1}{\lambda} & v^{-1} u^{-1} + u^{-1} + v + \frac{1}{\lambda}
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
  v^{-1} - v + u & \lambda v \\
  v^{-1} & u
\end{pmatrix},
\]

where $\lambda$ is a free spectral parameter.
All Trace First Integrals

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**Table:** # of new trace first integrals of degree $d$ generated from $\text{Tr}L^m$ that are not generated from $\text{Tr}L^k$, $k < m$. 
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The system

\[ u_\tau = Q_1(u, v, u^{-1}, v^{-1}), \quad v_\tau = Q_2(u, v, u^{-1}, v^{-1}) \quad (9) \]

is called an (infinitesimal) symmetry for

\[ u_t = P_1(u, v, u^{-1}, v^{-1}), \quad v_t = P_2(u, v, u^{-1}, v^{-1}) \quad (10) \]

if the flows (10) and (9) commute:

\[ u_{\tau t} = u_{t\tau}, \quad v_{\tau t} = v_{t\tau}. \]

The vector \((Q_1, Q_2)^t\) is called generator of the symmetry.
Symmetries II

Infinitely many symmetries $\rightarrow$ indicator of integrability
Symmetries II

Infinitely many symmetries → indicator of integrability

Example:

\[ u_\tau = -uvu - uv^2 + uv + (vu)^{-1} + v^{-2} + u^2 v^{-1} - uvu^{-1} \]
\[ + uv^{-2} + (vuv)^{-1} + u(vuv)^{-1}, \]
\[ v_\tau = vuv + vu^2 - vu - (uv)^{-1} - u^{-2} - v^2 u^{-1} + vuv^{-1} \]
\[ - vu^{-2} - (uvu)^{-1} - v(uvu)^{-1}. \]
Infinitely many symmetries $\rightarrow$ indicator of integrability

Example:

$$u_\tau = -uvu - uv^2 + uv + (vu)^{-1} + v^{-2} + u^2 v^{-1} - uvu^{-1} + uv^{-2} + (vuv)^{-1} + u(vuv)^{-1},$$

$$v_\tau = vuv + vu^2 - vu - (uv)^{-1} - u^{-2} - v^2 u^{-1} + vuv^{-1} - vu^{-2} - (uvu)^{-1} - v( uvu)^{-1}.$$ 

Conjecture: The dimension of the vector space $S_i$ of symmetries of degree $i$ is

$$\dim S_{4k} = 2k^2, \quad \dim S_{4k+1} = 2k^2 + 2k,$$

$$\dim S_{4k+2} = 2k^2 + 2k + 1, \quad \dim S_{4k+3} = 2k^2 + 4k + 1.$$
For $D_t$ defined by

$$D_t u = uv - uv^{-1} - v^{-1}, \quad D_t v = -vu + vu^{-1} + u^{-1}$$

find polynomials $Q_1, Q_2$ of $u, v, u^{-1}, v^{-1}$ such that

$$D_\tau u = Q_1, \quad D_\tau v = Q_2$$

commutes with $D_t$:

$$[D_t, D_\tau] u = 0, \quad [D_t, D_\tau] v = 0. \quad (11)$$
Direct Symmetry Computations II

Shorter necessary conditions:

As mentioned before, $I = uvu - 1$ is a first integral for KS and thus $I_k$ especially $I = vuv - 1$ are first integrals. Furthermore, $I_k, k = 0, \pm 1, \pm 2, ...$ are the only full first integrals up to degree 14.

A necessary condition for $D_\tau$ being a symmetry is $D_t D_\tau I = D_\tau D_t I = 0 \rightarrow D_\tau I = k_0 \sum_{k = -k_0} a_k I_k, k \in \mathbb{Z}$ and similarly $D_\tau I = k_0 \sum_{k = -k_0} b_k I_k, k \in \mathbb{Z}$ for sufficiently high $k_0$. These conditions involve extra unknown constants $a_k, b_k$ but these are first order conditions involving fewer terms than the full symmetry condition.
Direct Symmetry Computations II

Shorter necessary conditions:

As mentioned before $I = uvu^{-1}v^{-1}$ is a first integral for KS and thus $I^k$ especially $I^{-1} = vuv^{-1}u^{-1}$ are first integrals.
Direct Symmetry Computations II

Shorter necessary conditions:

As mentioned before \( I = uvu^{-1}v^{-1} \) is a first integral for KS and thus \( I^k \) especially \( I^{-1} = vuv^{-1}u^{-1} \) are first integrals.

Furthermore, \( I^k, \ k = 0, \pm 1, \pm 2, \ldots \) are the only full first integrals up to degree 14.
Direct Symmetry Computations II

Shorter necessary conditions:

As mentioned before $I = uvu^{-1}v^{-1}$ is a first integral for KS and thus $I^k$ especially $I^{-1} = vuv^{-1}u^{-1}$ are first integrals.

Furthermore, $I^k, k = 0, \pm 1, \pm 2, \ldots$ are the only full first integrals up to degree 14.

A necessary condition for $D_\tau$ being a symmetry is

$$D_t D_\tau I = D_\tau D_t I = 0 \rightarrow D_\tau I = \sum_{k=-k_0}^{k_0} a_k I^k, k \in \mathbb{Z}$$

and similarly

$$D_\tau I^{-1} = \sum_{k=-k_0}^{k_0} b_k I^k, k \in \mathbb{Z}$$

for sufficiently high $k_0$. These conditions involve extra unknown constants $a_k, b_k$ but these are first order conditions involving fewer terms than the full symmetry condition.
In utilizing the shortcut we calculated all symmetries, where $Q_i$ are polynomials of degree up to 16 in $u, v, u^{-1}, v^{-1}$. 
In utilizing the shortcut we calculated all symmetries, where $Q_i$ are polynomials of degree up to 16 in $u, v, u^{-1}, v^{-1}$.

The general ansatz for such a symmetry contains more than 172 million unknown coefficients satisfying over 1 billion linear conditions. The set of all such symmetries forms a commutative Lie algebra of dimension 32.
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The Algebra \( \mathcal{O} \) of local Operators

We will need to multiply from the left and the right, therefore the following definitions.

For any \( a \in M \) we denote by \( L_a \) and \( R_a \) the operators of left and right multiplications:

\[
L_a(b) = a \circ b, \quad R_a(b) = b \circ a.
\]

\( L_u, R_u, L_v, R_v \) generate the associative algebra \( \mathcal{O} \).
The Algebra $\mathcal{O}$ of local Operators

We will need to multiply from the left and the right, therefore the following definitions.

For any $a \in \mathcal{M}$ we denote by $L_a$ and $R_a$ the operators of left and right multiplications:

$$L_a(b) = a \circ b, \quad R_a(b) = b \circ a.$$ 

$L_u, R_u, L_v, R_v$ generate the associative algebra $\mathcal{O}$. 

For any element $a = a(x) \in \mathcal{M}$ the Frechet derivative $a_*$ is an $1 \times N$-matrix $a_*$ defined through:

$$
\frac{d}{d\epsilon} a(x + \epsilon \delta x)|_{\epsilon = 0} = a_*(\delta x). \tag{12}
$$
Frechet Derivative

For any element \( a = a(x) \in \mathcal{M} \) the Frechet derivative \( a_* \) is an \( 1 \times N \)-matrix \( a_* \) defined through:

\[
\frac{d}{d\epsilon} a(x + \epsilon \delta x)|_{\epsilon=0} = a_*(\delta x). \tag{12}
\]

For example, for \( h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1} \) we have

\[
h_*(\delta x) = \delta u - u^{-1}(\delta u)u^{-1} - u^{-1}(\delta u)u^{-1}v^{-1} + \delta v + ..
\]

\[
h_* = (1 - L_{u^{-1}}R_{u^{-1}} - L_{u^{-1}}R_{v^{-1}}R_{u^{-1}}, ..).
\]
For any element $a = a(x) \in \mathcal{M}$ the Frechet derivative $\mathbf{a}_*$ is an $1 \times N$-matrix $\mathbf{a}_*$ defined through:

$$
\frac{d}{d\epsilon} a(x + \epsilon \delta x)|_{\epsilon=0} = \mathbf{a}_*(\delta x) \quad (12)
$$

For example, for $h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1}$ we have

$$
\mathbf{h}_*(\delta x) = \delta u - u^{-1}(\delta u)u^{-1} - u^{-1}(\delta u)u^{-1}v^{-1} + \delta v + \ldots
$$

$$
\mathbf{h}_* = ( 1 - L_{u^{-1}}R_{u^{-1}} - L_{u^{-1}}R_{v^{-1}}R_{u^{-1}} , \ldots )
$$

In other words, if $u$ satisfies $u_t = \mathbf{F}$ then the $t-$derivative of any $a \in \mathcal{M}$ is given by

$$
D_t(a) = \mathbf{a}_*(\mathbf{F}) \quad (13)
$$
The gradient $\text{grad} (a)$ is the vector uniquely defined by:

$$\frac{d}{d\epsilon} a(x + \epsilon \delta x) |_{\epsilon=0} = a_*(\delta x) \sim \langle \delta x, \text{grad} (a(x)) \rangle.$$
The gradient $\text{grad} (a)$ is the vector uniquely defined by:

$$\frac{d}{d\epsilon} a(x + \epsilon \delta x) |_{\epsilon=0} = a_*(\delta x) \sim \langle \delta x, \text{grad} (a(x)) \rangle .$$

For example, for $h = u + v + u^{-1} + v^{-1} + u^{-1}v^{-1}$ we had

$$h_*(\delta x) = \delta u - u^{-1}(\delta u)u^{-1} - u^{-1}(\delta u)u^{-1}v^{-1} + \delta v + ..$$

and thus

$$\langle \delta x, \text{grad} h \rangle = (\delta u)(1 - u^{-2} - u^{-1}v^{-1}u^{-1}) + (\delta v)(...).$$
An adjoint operation $\star$ on $\text{Mat}_N(\mathcal{O})$ is uniquely defined by the quadratic form

$$
\langle (x, y), (x, y) \rangle = \text{Tr}(x^2) + \text{Tr}(y^2)
$$

and the formula

$$
\langle p, Q(q) \rangle \sim \langle Q^*(p), q \rangle, \quad (14)
$$

where $Q \in \text{Mat}_N(\mathcal{O})$, $p_i, q_i \in \mathcal{M}$. 
For example, to get $F^\star_*$ from 

$$F_\star = \begin{pmatrix}
R_v - R_{v^{-1}} & L_u + L_uL_{v^{-1}}R_{v^{-1}} + L_{v^{-1}}R_{v^{-1}} \\
-L_v - L_vL_{u^{-1}}R_{u^{-1}} - L_{u^{-1}}R_{u^{-1}} & -R_u + R_{u^{-1}}
\end{pmatrix}$$

one

- takes the transpose of $F_\star$,
- in each product reverses all L factors and all R factors,
- swaps $L \leftrightarrow R$
For example, to get $\mathbf{F}_\ast^*$ from

$$
\mathbf{F}_\ast = \begin{pmatrix}
R_v - R_v^{-1} & L_u + L_u L_v^{-1} R_v^{-1} + L_v^{-1} R_v^{-1} \\
-L_v - L_v L_u^{-1} R_u^{-1} - L_u^{-1} R_u^{-1} & -R_u + R_u^{-1}
\end{pmatrix}
$$

one

- takes the transpose of $\mathbf{F}_\ast$,
- in each product reverses all $L$ factors and all $R$ factors,
- swaps $L \leftrightarrow R$

and gets

$$
\mathbf{F}_\ast^* = \begin{pmatrix}
L_v - L_v^{-1} & -R_v - L_u^{-1} R_u^{-1} R_v - L_u^{-1} R_u^{-1} \\
R_u + L_v^{-1} R_v^{-1} R_u + L_v^{-1} R_v^{-1} & -L_u + L_u^{-1}
\end{pmatrix}
$$
A pre-Hamiltonian operator $\mathcal{P}$ is any $2 \times 2$ matrix that satisfies

$$\mathcal{P}_t = F_\star \mathcal{P} + \mathcal{P} F_\star^\ast$$

where $F_\star$ is the Frechet derivative (linearization) of the right hand side of the Kontsevich system and $F_\star^\ast$ stands for the adjoint of $F_\star$. 

A pre-Hamiltonian operator $\mathcal{P}$ is any $2 \times 2$ matrix that satisfies

$$\mathcal{P}_t = F_*\mathcal{P} + \mathcal{P}F_*^*$$

where $F_*$ is the Frechet derivative (linearization) of the right hand side of the Kontsevich system and $F_*^*$ stands for the adjoint of $F_*$. Pre-Hamiltonian operators map gradients of trace integrals like $\sigma = \text{Tr} \ (L^k_i)$ to symmetries.

$$(u, v)_T = \mathcal{P} (\text{grad}_u \sigma, \text{grad}_v \sigma)^T. \quad (15)$$
The operator

\[ P = \begin{pmatrix} R_u R_u - L_u L_u, & L_u L_v + L_u R_v - L_v R_u + R_u R_v \\ L_u R_v - L_v L_u - L_v R_u - R_v R_u, & L_v L_v - R_v R_v \end{pmatrix} \]

is a pre-Hamiltonian operator for KS.
Pre-Hamiltonian Operators II

The operator

\[ \mathcal{P} = \begin{pmatrix} R_u R_u - L_u L_u, & L_u L_v + L_v R_u - L_v R_u + R_u R_v \\ L_u R_v - L_v L_u - L_v R_u - R_v R_u, & L_v L_v - R_v R_v \end{pmatrix} \]

is a pre-Hamiltonian operator for KS.

Lax pair \( L, A \rightarrow \) trace integrals \( \text{Tr} \text{Tr} L^n \rightarrow \) pre-Hamiltonian \( \rightarrow \) symmetries
The operator

\[ \mathcal{P} = \begin{pmatrix}
R_u R_u - L_u L_u, & L_u L_v + L_u R_v - L_v R_u + R_u R_v \\
L_u R_v - L_v L_u - L_v R_u - R_v R_u, & L_v L_v - R_v R_v
\end{pmatrix} \]

is a pre-Hamiltonian operator for KS.

Lax pair \( L, A \rightarrow \) trace integrals \( \text{Tr Tr } L^n \) + pre-Hamiltonian \( \rightarrow \) symmetries

\( \rightarrow \) confirmation that the table of first integrals is complete (at least up to degree 14).
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Recursion Operators I

Recursion operators are operators that acting on a symmetry generate a new (higher degree) symmetry.
Recursion operators are operators that acting on a symmetry generate a new (higher degree) symmetry.

A $2 \times 2$ matrix $\mathcal{R}$, whose entries belong to $\mathcal{O}$ is called a recursion operator if for any symmetry generator $Q = (Q_1, Q_2)^t$ (therefore satisfying $D_t Q = F_*(Q)$) also the vector $\mathcal{R}Q$ is a symmetry generator and therefore satisfying

$$D_t(\mathcal{R}Q) = F_*(\mathcal{R}Q),$$

i.e.

$$(\mathcal{R}_t + \mathcal{R}F_* - F_*\mathcal{R})Q = 0$$

for any symmetry $Q$. 
Example: For

\[ u_t = uv, \quad v_t = -vu \]

a recursion operator is given by

\[ \mathcal{R} = \begin{pmatrix} R_u + R_v, & 0 \\ 0, & R_u + R_v \end{pmatrix}. \]
Example: For the Kontsevich system a recursion operator is given as follows.
Example: For the Kontsevich system a recursion operator is given as follows.

At first, a non-local vector \( \mathbf{B} = (B_1, B_2)^T \) is defined through

\[
P_\tau = [I, B_1]
\]
\[
\overline{P}_\tau = [I^{-1}, B_2]
\]

with \( I, P \) from previous slides and \( \overline{P} \) from involution \( u \leftrightarrow v \).
Recursion Operators IV

If $G$ defines a symmetry through

$$\frac{dx_\alpha}{d\tau} = G_\alpha(x_1, \ldots, x_N, x_1^{-1}, \ldots, x_N^{-1}), \quad G_\alpha \in \mathcal{M}$$

then a new symmetry of degree 2 higher than the degree of $G$ is given by

$$R_n B + R_l G$$

with $B$ from the previous slide and

$$R_n = \begin{pmatrix} L_u - R_u & L_u - R_u \\ L_v - R_v & L_v - R_v \end{pmatrix}$$

$$R_l = \begin{pmatrix} -R_v - R_u - L_{u-1} & L_{u-1}R_{v-1} - L_uL_{v-1}L_{u-1}(1 + R_{u-1} + R_v) + L_{u-1}R_{u-1}R_{v-1} - R_u \\ L_{u-1}L_{v-1}R_{v-1} - L_uL_{v-1}R_{v-1} - L_uL_{v-1}R_{v-1} + L_uL_{v-1} - L_{v-1}R_u \\ L_{v-1}L_{u-1}R_{u-1}R_v - L_vL_{u-1}R_{v-1} + L_vL_{u-1} - L_{u-1}R_v, & -R_u - R_{v-1}R_{u-1}R_{u-1} - L_vL_{u-1}L_{v-1}(1 + R_{v-1} + R_u) + L_{v-1}R_{v-1}R_{u-1}R_u \end{pmatrix}$$
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After a proper scaling of $u$, $v$ and $t$ the Kontsevich system can be rewritten as

$$u_t = uv - \varepsilon^2 uv^{-1} - \varepsilon^3 v^{-1}, \quad v_t = -vu + \varepsilon^2 vu^{-1} + \varepsilon^3 u^{-1}. \quad (16)$$

This system can be regarded as a Laurent deformation of the Mikhailov-Sokolov system

$$u_t = uv, \quad v_t = -vu.$$
After a proper scaling of \( u, v \) and \( t \) the Kontsevich system can be rewritten as

\[
\begin{align*}
    u_t &= u v - \varepsilon^2 u v^{-1} - \varepsilon^3 v^{-1}, \\
    v_t &= -v u + \varepsilon^2 v u^{-1} + \varepsilon^3 u^{-1}.
\end{align*}
\] (16)

This system can be regarded as a Laurent deformation of the Mikhailov-Sokolov system

\[
    u_t = u v, \quad v_t = -v u.
\]

We found several more Laurent deformations of this system having infinitesimal symmetries.
Further Laurent Deformations

**Theorem.** All systems of the form:

\[ u_t = uv + P_1(u, u^{-1}, v, v^{-1}), \quad v_t = -vu + Q_1(u, u^{-1}, v, v^{-1}), \]

where \( P_1, Q_1 \) are the most general inhomogeneous degree 2 polynomials without quadratic term in \( u, v \)
Further Laurent Deformations

**Theorem.** All systems of the form:

\[ u_t = uv + P_1(u, u^{-1}, v, v^{-1}), \quad v_t = -vu + Q_1(u, u^{-1}, v, v^{-1}), \]

where \( P_1, Q_1 \) are the most general inhomogeneous degree 2 polynomials without quadratic term in \( u, v \) which have commuting flows of the form

\[ u_\tau = P_2(u, u^{-1}, v, v^{-1}), \quad v_\tau = Q_2(u, u^{-1}, v, v^{-1}), \]

where \( P_2, Q_2 \) are the most general inhomogeneous degree 4 polynomials,
Further Laurent Deformations

**Theorem.** All systems of the form:

\[ u_t = uv + P_1(u, u^{-1}, v, v^{-1}), \quad v_t = -vu + Q_1(u, u^{-1}, v, v^{-1}), \]

where \( P_1, Q_1 \) are the most general inhomogeneous degree 2 polynomials without quadratic term in \( u, v \) which have commuting flows of the form

\[ u_\tau = P_2(u, u^{-1}, v, v^{-1}), \quad v_\tau = Q_2(u, u^{-1}, v, v^{-1}), \]

where \( P_2, Q_2 \) are the most general inhomogeneous degree 4 polynomials, are shown in the following list:
A Listing of Laurent Deformations I

\begin{align*}
  u_t &= uv + a_1 uv^{-1} + a_2 v^{-1} \\
  v_t &= -vu + b_1 u^{-1} v - a_2 u^{-1}
\end{align*}

\begin{align*}
  u_t &= uv + a_1 v^{-1} u \\
  v_t &= -vu + a_2 u^{-1} v,
\end{align*}

\begin{align*}
  u_t &= uv - a_1 v^{-1} u + a_1 \\
  v_t &= -vu + a_1 u^{-1} v - a_1,
\end{align*}

\begin{align*}
  u_t &= uv + a_1 uv^{-1} + b_1 \\
  v_t &= -vu + b_1 vu^{-1} + a_1,
\end{align*}
A Listing of Laurent Deformations II

\begin{align*}
u_t &= uv + a_1 uv^{-1} + a_2 u + a_3 \\
v_t &= -vu - a_2 u + a_3 vu^{-1} + b_1 u^{-1} v - a_1 v^{-1} u + (a_2 a_3 + a_2 b_1) u^{-1} \\
    &\quad + (a_1 a_3 + a_1 b_1) u^{-1} v^{-1} + (a_2^2 + a_3 b_1) u^{-2} + b_2,
\end{align*}

\begin{align*}
u_t &= uv + a_1 u + a_2 v + a_3 \\
v_t &= -vu - a_1 u - a_2 v - a_3,
\end{align*}

\begin{align*}
u_t &= uv + 2a_1 uv^{-1} - a_1 v^{-1} u - a_1 \\
v_t &= -vu - 2a_1 vu^{-1} + a_1 u^{-1} v + a_1,
\end{align*}

\begin{align*}
u_t &= uv + a_1 uv^{-1} - a_2 v^{-1} + a_3 u \\
v_t &= -vu + b_1 vu^{-1} + a_2 u^{-1} + b_2 v,
\end{align*}
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Computational Problems

We have two classes of problems.

1. Integrability of a given equation.
Computational Problems

We have two classes of problems.

1. Integrability of a given equation.
   Computing first integrals $\rightarrow$ large linear systems.

2. Classification problem.
   In this case we need to solve bi-linear algebraic systems to find systems with symmetries, both at the same time.
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   Computing first integrals $\rightarrow$ large linear systems.
   Searching for a Lax-pair $\rightarrow$ bi-linear inhomogeneous systems.

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   In this case we need to solve bi-linear algebraic systems to find systems with symmetries, both at the same time.
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1. Integrability of a given equation.
   Computing first integrals $\rightarrow$ large linear systems.
   Searching for a Lax-pair $\rightarrow$ bi-linear inhomogeneous systems.
   Identifying $L, A$ pairs through conjugation $\rightarrow$ many medium linear systems.

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   Computing first integrals → large linear systems.
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   Computing symmetries → large linear systems.
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   Computing a pre-Hamiltonian structure $\rightarrow$ small linear systems.

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Computational Problems

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   Computing a pre-Hamiltonian structure $\rightarrow$ small linear systems.
   Computation of recursion operator $\rightarrow$ medium linear systems.

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Computational Problems

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   Searching for a Lax-pair $\rightarrow$ bi-linear inhomogeneous systems.
   Identifying $L, A$ pairs through conjugation $\rightarrow$ many medium linear systems.
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   Computation of recursion operator $\rightarrow$ medium linear systems.
   Application of recursion operator $\rightarrow$ medium linear systems.

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1. Integrability of a given equation.
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Computational Problems

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Direct Symmetry Computations, the Conditions:

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<td>78,730</td>
<td>104,974</td>
<td>157,392</td>
<td>409,470</td>
<td>1,623,842</td>
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<td>314,926</td>
<td>472,312</td>
<td>1,258,526</td>
<td>5,304,562</td>
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<tr>
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<td>1,417,088</td>
<td>3,862,086</td>
<td>17,212,778</td>
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<tr>
<td>12</td>
<td>2,125,762</td>
<td>2,834,350</td>
<td>4,251,432</td>
<td>11,835,758</td>
<td>55,535,578</td>
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<tr>
<td>13</td>
<td>6,377,290</td>
<td>8,503,054</td>
<td>12,754,480</td>
<td>36,228,892</td>
<td>178,298,450</td>
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<tr>
<td>14</td>
<td>19,131,874</td>
<td>25,509,166</td>
<td>38,263,640</td>
<td>110,777,292</td>
<td>569,970,466</td>
<td>25</td>
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<tr>
<td>15</td>
<td>57,395,626</td>
<td>76,527,502</td>
<td>$&gt; 1.1 \times 10^8$</td>
<td>$&gt; 3.3 \times 10^8$</td>
<td>$&gt; 1.7 \times 10^9$</td>
<td>31</td>
</tr>
<tr>
<td>16</td>
<td>$&gt; 172 \times 10^6$</td>
<td>$&gt; 229 \times 10^6$</td>
<td>$&gt; 3.4 \times 10^8$</td>
<td>$&gt; 1 \times 10^9$</td>
<td>$&gt; 5.7 \times 10^9$</td>
<td>32</td>
</tr>
</tbody>
</table>

Table: For a symmetry ansatz of degree $N$ are listed the $\# \nu$ of variables, the $\# e_1$ of equations and $\# t_1$ of terms of system $D_\tau (I) = 0$ and the $\# e_2$ of equations and $\# t_2$ of terms of system $[D_t, D_\tau](u, v) = 0$ and the $\# p$ of free parameters of the solution.
## Quantitative Progress

<table>
<thead>
<tr>
<th>N</th>
<th>stream solve</th>
<th>sorting by size, stream solve</th>
<th>1-term equ. (A), sorting by size, stream solve</th>
<th>1-term equ. (B), sorting by size, stream solve</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>solve subst.</td>
<td>solve subst.</td>
<td>solve subst.</td>
<td>solve subst.</td>
</tr>
<tr>
<td>3</td>
<td>.02</td>
<td>.01</td>
<td>.01</td>
<td>.00</td>
</tr>
<tr>
<td>4</td>
<td>.14</td>
<td>.01</td>
<td>.15</td>
<td>.02</td>
</tr>
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<td>5</td>
<td>1.58</td>
<td>.17</td>
<td>1.29</td>
<td>.23</td>
</tr>
<tr>
<td>6</td>
<td>20.79</td>
<td>1.46</td>
<td>15.30</td>
<td>2.36</td>
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<td>205.73</td>
<td>8.97</td>
<td>85.16</td>
<td>9.00</td>
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<td>82.50</td>
<td>807.62</td>
<td>92.77</td>
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<tr>
<td>9</td>
<td>53,935.88</td>
<td>1,191.47</td>
<td>13,335.13</td>
<td>1,481.95</td>
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<tr>
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<tr>
<td>11</td>
<td></td>
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<tr>
<td>12</td>
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<tr>
<td>13</td>
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</table>

**Table:** Times in sec for solving the symmetry conditions $[D_t, D_\tau](u, v) = 0$ for each degree $N$ by different methods
Comparison with other Programs

<table>
<thead>
<tr>
<th>N</th>
<th></th>
<th>Maple 14</th>
<th></th>
<th>LinBox</th>
<th></th>
<th>LinSolve (Reduce)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>solve</td>
<td>total</td>
<td>default</td>
<td>sparse</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sym+Fi</td>
<td>Sym</td>
<td>Sym+Fi</td>
<td>Sym</td>
<td>Sym+Fi</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>.03</td>
<td>.03</td>
<td>.06</td>
<td>.06</td>
<td>.00</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>.09</td>
<td>.10</td>
<td>.19</td>
<td>.22</td>
<td>.02</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>.30</td>
<td>.31</td>
<td>.63</td>
<td>.70</td>
<td>.12</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>1.07</td>
<td>.96</td>
<td>7.81</td>
<td>2.04</td>
<td>30.6</td>
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<tr>
<td>7</td>
<td></td>
<td>6.01</td>
<td>6.88</td>
<td>11.02</td>
<td>13.81</td>
<td>14.9</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>21.55</td>
<td>17.23</td>
<td>47.28</td>
<td>35.65</td>
<td>3080</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>78.34</td>
<td>69.08</td>
<td>154.40</td>
<td>131.90</td>
<td>2318</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>312.3</td>
<td>273.1</td>
<td>587.6</td>
<td>508.4</td>
<td>21610</td>
</tr>
<tr>
<td>11</td>
<td></td>
<td>1237</td>
<td>1127</td>
<td>2262</td>
<td>2015</td>
<td>21210</td>
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<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
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<td>26.67</td>
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<td>13</td>
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<td>91.27</td>
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<td>402.70</td>
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</table>

**Table:** Times in sec for solving the symmetry conditions $[D_t, D_\tau](u, v) = 0$ for each degree $N$ by different programs
## Iterating the Formulation and Solution of Systems I

<table>
<thead>
<tr>
<th></th>
<th>([D_t, D_\tau](u, v) = 0) at once</th>
<th>([D_t, D_\tau](u, v) = 0) in two stages</th>
<th>(D_\tau(I) = 0) first</th>
<th>Iteration first</th>
</tr>
</thead>
<tbody>
<tr>
<td>formulation of ansatz for (D_\tau)</td>
<td>61</td>
<td>61</td>
<td>61</td>
<td>61</td>
</tr>
<tr>
<td>computation of (D_\tau^{-1})</td>
<td>122</td>
<td>122</td>
<td>122</td>
<td>122</td>
</tr>
<tr>
<td>5 × computing + splitting (D_\tau(I) = 0), (<a href="u">D_t, D_\tau</a> = 0), (D_\tau(I) = 0), (<a href="v">D_t, D_\tau</a> = 0), each time only extracting and using 1-term conditions</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>587</td>
</tr>
<tr>
<td>computation and splitting of (D_\tau(I) = 0), extracting and using 1-term conditions</td>
<td>–</td>
<td>–</td>
<td>391</td>
<td>1</td>
</tr>
<tr>
<td>computation and splitting of (<a href="u">D_t, D_\tau</a> = 0)</td>
<td>1216</td>
<td>1216</td>
<td>235</td>
<td>2</td>
</tr>
<tr>
<td>extracting and using 1-term conditions, keeping others</td>
<td>–</td>
<td>536</td>
<td>22</td>
<td>1</td>
</tr>
<tr>
<td>computation and splitting of (<a href="v">D_t, D_\tau</a> = 0)</td>
<td>1216</td>
<td>63</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>complete solution of all remaining equations</td>
<td>365</td>
<td>253</td>
<td>241</td>
<td>216</td>
</tr>
</tbody>
</table>

**Table:** Times in sec of the whole symmetry computation for \(N = 13\)
(continuation of the table above:)

<table>
<thead>
<tr>
<th></th>
<th>$[D_t, D_\tau](u, v) = 0$ at once</th>
<th>$[D_t, D_\tau](u, v) = 0$ in two stages</th>
<th>$D_\tau(I) = 0$ first</th>
<th>Iteration first</th>
</tr>
</thead>
<tbody>
<tr>
<td>substitution of solution in $D_\tau(u, v)$</td>
<td>57</td>
<td>33</td>
<td>31</td>
<td>26</td>
</tr>
<tr>
<td>total CPU time</td>
<td>3037</td>
<td>2284</td>
<td>1105</td>
<td>1018</td>
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<tr>
<td>total garbage collection time</td>
<td>578</td>
<td>210</td>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>overall time</td>
<td>3615</td>
<td>2494</td>
<td>1114</td>
<td>1034</td>
</tr>
</tbody>
</table>

**Table:** Times in sec of the whole symmetry computation for $N = 13$
Outline

Non-commutative ODEs
First Integrals and Lax Pairs
Symmetries
Pre-Hamiltonian Operators
Recursion Operators
Classifications
Computer Runs
References
References I

Integrability, non-abelian ODE Systems

- M. Kontsevich *Noncommutative identities* arXiv:1109.2469v1
References II

**Computer Algebra**


**Compatible Poisson Brackets**

The End

Thank you!