Classification of integrable super-systems using the SsTools environment ⋆

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Abstract

A classification problem is proposed for supersymmetric evolutionary PDE that satisfy the assumptions of nonlinearity, nondegeneracy, and homogeneity. Four classes of nonlinear coupled boson-fermion systems are discovered under the weighting assumption $|f| = |b| = |D_t| = \frac{1}{2}$. The syntax of the REDUCE package SSTOOLS, which was used for intermediate computations, and the applicability of its procedures to the calculus of super-PDE are described.

Key words: Integrable super-systems, symmetries, recursions, classification, symbolic computation, REDUCE
PACS: 02.30.Ik, 02.70.Wz, 12.60.Jv
1991 MSC: [2000] 35Q53, 37K05, 37K10, 81T40

PROGRAM SUMMARY

Title of program: SSTOOLS

Catalogue number: `xxxx`

Program obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland; see also [1]

⋆ *Subject classification*: Computational models in physics (Symmetry integrability), Programming environment (Symbolic and geometric methods).

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Licensing provisions: none

Computer for which the program is designed and others on which it has been tested:
Computers: (i) IBM PC, (ii) cluster
Installations: (i) Brock University, St. Catharines, Ontario, Canada L2S 3A1; (ii) SHARCNET `http://www.sharcnet.ca`

Operating system under which the program has been tested: LINUX

Programming language used: REDUCE 3.7, REDUCE 3.8

Memory required to execute with typical data: problem dependent (10 Mb – 1Gb), typical working size < 100 Mb

No. of bits in a word: $32, 64$

No. of processors used: (i) 1; (ii) multiple

Peripherals used: no

Has the code been vectorized? no

No. of lines in distributed program including documentation file, test data, etc.: 2485

Distribution format: ASCII

Online access and tutorials: `http://lie.math.brocku.ca/crack/susy/`

Keywords: Integrable super-systems; symmetries; recursions; classification; symbolic computation; REDUCE

Nature of physical problem: The program allows the classification of $N \geq 1$ supersymmetric nonlinear scaling-invariant evolution equations $\{f_t = \varphi^f, b_t = \varphi^b\}$ that admit infinitely many local symmetries propagated by recursion operators; here $b(x,t;\theta)$ is the set of bosonic super-fields and $f(x,t;\theta)$ are fermionic super-fields.

Method of solution: First, (half-)integer weights $|f|$, $|b|$, …, $|D_t|$, $|D_x| \equiv 1$ are assigned to all variables and derivatives and then pairs of commuting flows that are homogeneous w.r.t. these weights are constructed. Secondly, the seeds of higher symmetry sequences [2] for the systems are sorted out, and finally the recursion operators that generate the symmetries are obtained [3]. The intermediate algebraic systems upon the undetermined coefficients are solved by using [4].

Restrictions on the complexity of the problem: Computation of symmetries of high differential order for very large evolutionary systems may cause memory restrictions. Additional size/time restrictions may occur if the homogeneity weights of some super-fields are non-positive, see section 1.2 of the Long Write-Up.

Typical running time: depends on the size and complexity of the input system and varies between seconds and minutes.

Unusual features of the program: SsTools has been extensively tested using hundreds of PDE systems within three years on UNIX-based PC-machines. SsTools is applicable to the computation of symmetries, conservation laws, and Hamiltonian structures for $N \geq 1$ evolutionary super-systems with any N. SsTools is also useful for performing extensive arithmetic of general nature including differentiations of super-field expressions.

References:

[1] `http://lie.math.brocku.ca/crack/susy/sstools.red`; The support package Crack is obtained from `http://lie.math.brocku.ca/crack/src/crack.tar.gz`

[2] P.J. Olver, Applications of Lie groups to differential equations, 2nd ed., Springer, Berlin, 1993.

[3] I. S. Krasil'shchik, P. H. M. Kersten, Symmetries and recursion operators for classical and supersymmetric differential equations, Kluwer, Dordrecht, 2000.

[4] T. Wolf, Applications of Crack in the classification of integrable systems, CRM Proc. Lecture Notes 37 (2004), 283–300.

LONG WRITE–UP

Introduction

The principle of symmetry belongs to the foundations of modern mathematical physics. The differential equations that constitute *integrable* [1] models practically always admit symmetry transformations and, reciprocally, new classes of integrable phenomena in physics are obtained by postulating some symmetry invariance. The presence of symmetry transformations in a system yields two types of explicit solutions: those which are invariant under a transformation (sub)group and, secondly, the solutions obtained by propagating a known solution by the same group. The two schemes for constructing new solutions of PDE are crucial for systems of super-equations of mathematical physics (e.g., supergravity models); these equations involve commuting (bosonic, or 'even') and anticommuting (fermionic, or 'odd') independent variables and/or unknown functions. Indeed, other methods for solving nonlinear equations need a special adaptation to the super-field setting, see e.g. [2]; another approach to integrability of supersymmetric equations has been investigated in [3]. Hence the symmetry considerations [1], which are based on the computation of infinitesimal symmetry generators for PDE, become highly important. The arising computational problems are unmanageable without computer algebra that permits handling relevant systems.

In the literature it has been observed (see [1,4] and references therein) that the principal phenomena in nature are governed by systems of PDE that admit higher symmetries, that is, the symmetry transformations that involve higher order derivatives of the unknown functions. Simultaneously, when dealing with supersymmetric models of theoretical physics, it is often hard to predict whether a certain mathematical approximation will be truly integrable or not. Therefore we apply the symbolic computational approach to the physical problem of classifying the systems that exhibit necessary integrability features. It must be noted that the very idea to filter out integrable cases using the presence of 'many' symmetries is widely accepted in the computer branch of modern mathematical physics, see e.g. [4,5]. These systems are called symmetry integrable [1,4]); in some cases, they can be transformed to exactly solvable equations or their extensions. In view of this classification task, we analyzed fermionic extensions of the Burgers and Boussinesq equations and related the former with evolutionary systems on associative algebras in [6].

This paper is organized as follows. First we formulate the axioms of the classification problem for $N = 1$ supersymmetric systems of evolutionary PDE. In sections 1.1 and 1.2 we describe the two modes of the procedure `ssym` in the package SsTools that allow, respectively, finding unordered pairs of commuting flows and the computation of symmetries for previously found systems. Also, in section 1.2 we review a geometric (coordinate-free) algorithm for constructing recursion operators. However, we refer to [5,7] for basic notions and concepts in the geometry of (super)PDE, see also [8–10,16] and references therein. The principal result of this paper is that there exist only four nonlinear coupled boson-fermion systems that satisfy the axioms and the weighting $|f| = |b| = |D_t| = \frac{1}{2}$. Section 1 concludes with the classification of their recursion operators. In the following sections we investigate properties of these four systems, giving simultaneously the syntax and describing applicability of SsTools subroutines developed for the calculation of the scaling weights, symmetries, linearizations, conservation laws needed for introducing the nonlocalities, and recursion operators. Finally, sample runs of SsTools are given.

1 The classification problem

Let us introduce some notation. We denote by θ the super-variable and we put $\mathcal{D} \equiv D_\theta + \theta\, D_x$ such that $\mathcal{D}^2 = D_x$ and $[\mathcal{D}, \mathcal{D}] = 2D_x$; here D_θ and D_x are the derivatives w.r.t. θ and x, respectively. Throughout this text, the operator $\mathcal{D}$ acts on the succeeding super-field unless stated otherwise explicitly.

The package SsTools in Reduce (ver. 3.7 or higher) is designed for calculus on super-PDE. It was implemented for solving the classification problem for symmetry integrable $N \geq 1$ super-PDE and fermionic extensions of the purely bosonic evolution equations. Compared to other programs listed in the overview [11], it is able to work with (super)PDE and compared to programs mentioned in [12–14] the package SsTools has not only the ability to compute higher symmetries for a given system but also to solve the non-linear problem of computing polynomial systems and their higher symmetries *at once* if certain homogeneity weights are specified; here we also note the program [15] for computations performed in presence of the supersymmetry. In SsTools, the following list of seven axioms was postulated:

(1) each system of equations $\binom{f_t}{b_t}$ admits at least one higher symmetry $\binom{f_s}{b_s}$;
(2) all equations are spatial translation invariant and do not depend on the time t explicitly;
(3) none of the evolution equations involves only one field and hence none of the right-hand sides vanishes;
(4) at least one of the right-hand sides in either the evolution equation or its symmetry is nonlinear;
(5) at least one equation in a system or at least one component of its symmetry contains a fermionic field or the super-derivative $\mathcal{D}$;
(6) the equations are scaling invariant: their right-hand sides are differential polynomials homogeneous with respect to a set of (half-)integer weights [1] $|\mathcal{D}| \equiv \frac{1}{2}$, $|D_x| \equiv 1$, $|D_t| > 0$, $|f|$, $|b| > 0$; we also assume that the positive weight of D_s that occurs through $\binom{f_s}{b_s}$ is (half-)integer.

The time t and the parameter s along the integral trajectories of the symmetry fields can either assign the bosonic flows b_t, b_s to b and the fermionic flows f_t, f_s to f or reverse the parities, that is, $b_{\bar{t}}$ or $b_{\bar{s}}$ become fermionic and $f_{\bar{t}}$ or $f_{\bar{s}}$ are bosonic. Here we use the notation $\bar{t}$ or $\bar{s}$ if the parities of $f_{\bar{t}}$, $b_{\bar{t}}$ or $f_{\bar{s}}$, $b_{\bar{s}}$ are opposite to the parities of f and b, respectively.

[1] The *weights* $|\cdot|$ take all variables (e.g., a fermionic super-field f) and derivations (e.g., $D_t \equiv \mathrm{d}/\mathrm{d}t$) to (half-)integers ($|f|$ and $|D_t| \in \frac{1}{2}\mathbb{Z}$, respectively) such that the weight of any product is the sum of weights of the factors (hence $|f_t| = |D_t(f)| = |D_t| + |f|$); only homogeneous algebraic expressions with equal weights of the components are considered from now on.

The procedure `ssym` for computing symmetries, which is the central procedure [2] of the SsTools package, can be used in two modes. The first mode of `ssym` developed together with W. Neun was used for finding $N = 1$ supersymmetric coupled boson-fermion evolutionary systems that satisfy the above axioms. The axioms are incorporated in the ansatz generators. Hence, having the weights $|f|$, $|b|$, $|t|$ or $|\bar{t}|$, and $|s|$ or $|\bar{s}|$ specified, we face the problem of constructing a pair of commuting flows $(f, b)_{t(\text{or } \bar{t})}$ and $(f, b)_{s(\text{or } \bar{s})}$. The procedure `ssym` generates the homogeneous ansatz with undetermined coefficients in both flows, and calls the program Crack [16] for solving automatically the overdetermined algebraic systems. The standard call in the first mode is

```
ssym(N,tw,sw,afwlist,abwlist,eqnlist,fl,inelist,flags);
```

where

`N`	... the number of superfields θ^i;
`tw`	... 2 times the weight of D_t;
`sw`	... 2 times the weight of D_s;
`afwlist`	... list of weights of the fermion fields `f(1)`, `f(2)`,..., `f(nf)`;
`abwlist`	... list of weights of the boson fields `b(1)`, `b(2)`,..., `b(nb)`;
`eqnlist`	... list of extra conditions on the undetermined coefficients;
`fl`	... extra unknowns in `eqnlist` to be determined;
`inelist`	... a list, each element of it is a non-zero expression or a list with at least one of its elements being non-zero;
`flags`	... `init`: only initialization of global data, `zerocoeff`: all coefficients $= 0$ which do not appear in `inelist`, `tpar`: if the time variable t changes parity, `spar`: if the symmetry variable s changes parity.

The nonnegative integer N is the number of super-fields θ^i such that $\mathcal{D}_i \equiv D_{\theta^i} + \theta^i D_x$ are the respective super-derivatives, $\mathcal{D}_i^2 = D_x$. The computer notation for $\mathcal{D}_i$ is `d(i,...)`, thus opening the opportunity to work with $N \geq 1$ under SsTools doing the calculus of super-fields; here we also note that the syntax of D_x is `df(..,x)` and similarly for D_t. In addition, SsTools permits the investigation of non-supersymmetric systems (with $N = 0$, which is not needed for the four systems within the classification below). What may be of interest to the specialists working with super-calculus and super-PDE is that after loading SsTools the simplification of polynomial expressions

[2] The command `sshelp();` provides a detailed user's guide on the package including examples that illustrate admissible combinations of the flags in the procedure calls. All package subroutines are described in this paper where appropriate. Online access to SsTools without any need to install the computer algebra system is provided through `http://lie.math.brocku.ca/crack/susy`.

which involve the anti-commuting derivations $\mathcal{D}_i$ and fields f^i is performed automatically, taking care of all the minus signs that arise in multiplications and derivatives.

The numbers `nf` of fermion fields and `nb` of boson fields are determined automatically through the number of elements in the lists `afwlist` and `abwlist`. The input list `eqnlist` can contain substitutions, like `p2=1`, or expressions which are to be set to zero, like `p3*r4+p2*r3`. Conditions as `p2=1` are executed instantly at the time of formulating the ansatz for the pair of commuting flows, while expressions without `=` sign are added as conditions `...=0` to the other equations when calling CRACK. Typically, one would run the program first with the flag `init` to see which ansatz for the system and its symmetry is generated and then start `ssym` again without `init` but with the option to add extra conditions on the unknown coefficients either through the flag `zerocoeff` or through specific extra conditions in `eqnlist` or entries in `inelist`.

Using the bounds $0 < |f|, |b| \leq 5$ and $0 < |D_t| < |D_s| \leq 5$, we constructed the experimental database [17], which contains 1830 equations such that $N \leq 2$ (the duplicate PDEs that appeared owing to possible non-uniqueness of the weights are not counted) and their 4153 symmetries (plus the translations along x and t, and plus the scalings whose number is in fact infinite). In this paper we investigate properties of the systems for which the weights of the times t, $\bar{t}$ are $|t| = |\bar{t}| = -\frac{1}{2} \Leftrightarrow |D_t| = |D_{\bar{t}}| = \frac{1}{2}$, assuming further that $|f| = |b| = \frac{1}{2}$ (the weights may not be uniquely defined). Having fixed the weights and using SSTOOLS, we discovered four systems that satisfy the set of axioms; they are classified as follows, see Table 1 on p. 12 below. System (2) is related to the fermionic extension of the Burgers equation [6]. Also, we get a super-field representation (4) of the Burgers equation (5) and discuss its properties which are revealed only by the presence of super-variables. Further, there is a multiplet of 14 systems (9) with the parity preserving time t; these systems possess s-symmetries and higher $\bar{s}$-supersymmetry flows. Finally, we obtain a unique system (11) which has the parity reversing times $\bar{t}$ and $\bar{s}$; using this system, we illustrate the process of calculating the recursion operators in the test output.

1.2 Reconstruction of recursion operators

The second application of SSTOOLS is the construction of symmetries, conservation laws, and recursion operators for the symmetry algebras of *given* systems, i.e. systems $\mathcal{E}$ that have been obtained before by `ssym` and placed in the database [17]. The method of Cartan's forms [5] for the recursion operators is applied, see [6] for examples based on the notation introduced below. Within this approach, the recursions $\mathcal{R}$ are regarded as symmetries $F_{s_R} = \mathcal{R}^f$,

$B_{s_R} = \mathcal{R}^b$ of the linearized equations Lin $\mathcal{E}$. Namely, we ignore the differential function structure of the symmetry flows $f_s = F$, $b_s = B$ and regard F and B as the components of solutions of the linearized equations Lin $\mathcal{E}$, which are defined in the next paragraph. The expressions $\mathcal{R} = R(F, B)$ are recursion operators for $\mathcal{E}$ if each $\mathcal{R}$ satisfies the linearized equation Lin $\mathcal{E}$ again and if they are linear w.r.t. F, B, and their derivatives.

The understanding of linearized systems Lin $\mathcal{E}$ from a computational viewpoint is as follows; we consider the differential polynomial case since this is what we and SsTools deal with. Given a system $\mathcal{E}$, formally assign the new 'linearized' fields $F^i =$ `f(nf+i)` and $B^j =$ `b(nb+j)` to $f^i =$ `f(i)` and $b^j =$ `b(j)`, respectively, with $i = 1,\ldots,$`nf` and $j = 1,\ldots,$`nb`. Pass through all equations, and whenever a power of a derivative of a variable f^i or b^j is met, differentiate (in the usual sense) this power with respect to its base, multiply the result from the right by the same order derivative of F^i or B^j, respectively, and insert the product in the position where the power of the derivative was met. Now proceed by the Leibnitz rule. The final result, when all equations in the system $\mathcal{E}$ are processed, is the linearized system Lin $\mathcal{E}$. For example, the linearized counterpart of $b_t = b(\mathcal{D}f)^2$ is $B_t = B \cdot (\mathcal{D}f)^2 + 2b\mathcal{D}f \cdot \mathcal{D}F$.

The linearization Lin $\mathcal{E}$ for a system of evolution equations $\mathcal{E}$ is obtained using the procedure `linearize`:

```
linearize(pdes,nf,nb);
```

where
`pdes` ... list of equations with first order derivatives in l.h.s.;
`nf` ... number of the fermion fields `f(1)`,...,`f(nf)`;
`nb` ... number of the boson fields `b(1)`,...,`b(nb)`.

The values `nf` and `nb` may be greater than one for the coupled boson-fermion systems with unique f and b (only such systems are studied throughout this paper) because some additional variables (the nonlocalities, see (1) for a detailed explanation) can have been added previously by hand using the method of trivializing conservation laws. The actual number of fields is doubled by a call of `linearize`, and in the sequel we always denote these new 'linearization variables' by the respective capital letters.

Now we describe the syntax of a call in the second mode of using procedure `ssym`. In this mode symmetries are computed (which can be interpreted later as recursion operators) for a given evolutionary system `pdes`. The call is

```
depend {f(1),f(2),..,f(nf),b(1),b(2),..,b(nb)},x,t;
max_deg:=..;
hom_wei:={{sw_1,afwlist_1,abwlist_1},..};
ssym(N,tw,sw,afwlist,abwlist,pdes,fl,inelist,flags);
```

where this time we use the actual numbers of fields (they may have increased after `linearize` produced a bigger system by assigning new F^i to f^i and B^j to b^j). We now have

`pdes` ... the list of equations `df(...,t)=...` specifying the (linearized) system for which symmetries are to be computed, as well as substitution rules `...=>...` and other conditions `...=...` (see Remark 1 below);

`flags` ... the list of flags now includes:
`lin`: the symmetry to be determined is linear w.r.t. the fields `f(i)`, `b(j)` such that $i > \texttt{nf}/2$ and $j > \texttt{nb}/2$;
`filter`: if present this flag indicates that the homogeneities listed in `hom_wei` have to be used in addition;
`tpar`, `spar`: $D_{\bar{t}}$ and $D_{\bar{s}}$, respectively, is parity changing.

If the right-hand sides in `pdes` involve any constants which should be computed such that the higher symmetries do exist, then these constants are listed in the input list `fl`, like `{p1,p2,...}` otherwise `fl` is `{}`. This may be needed when we suspect the presence of a multiplet of systems parameterized by discrete admissible values of these coefficients `p1,p2,...`. If the constant coefficients are not given in the list `fl`, then they are treated as independent parameters and only symmetries (and not these constants) are determined for generic values of the constant coefficients.

Next, if a boson's weight is non-positive or a fermion's weight is negative then the global variable `max_deg` must have a positive integer value, which is the highest power of such fields and their derivatives in any ansatz that the program will generate.

Remark 1 The key for procedure `ssym` to know whether to compute symmetries for the *given* system (this is mode 2) or to make an ansatz for the *yet unknown* system with a symmetry (hence, for a pair of commuting flows within mode 1) is the first element of the input list `pdes`. If this is of type `...=...` or of type `...=>...` and if the left hand side is `df(...)` or `d(...)`, then:

- All elements of `pdes` of the form `df(...,t)=...` are collected and interpreted as the system for which symmetries are to be computed.
- All other elements of `pdes` of the form `...=>...` are assumed to have `df(...)` or `d(...)` on the left-hand side. These substitution rules and differential consequences of them are used to simplify right-hand sides of the system, in making the ansatz for the symmetry, and in further computations.
- All other elements of type `...=...` or elements that are simple expressions (which are set to 0) are interpreted as extra conditions on the undetermined coefficients.

There are two types of relations to be provided in `pdes` which is the reason why substitutions `=>` are needed in addition to the usual relations `=`. First, the very algorithm [5,10] for finding recursions of symmetry algebras of equations $\mathcal{E}$ suggests to treat the recursions as symmetries of the linearized systems $\mathrm{Lin}\,\mathcal{E}$, taking into account the original equations (that is, replacing their left-hand sides with their right-hand sides) and differential consequences of them. The linearized system, which involves time derivatives of Cartan's forms, is included in the list `pdes` in form of relations `df(f(i),t)=`..., $i >$ `nf`/2 and `df(b(j),t)=`..., $j >$ `nb`/2.

Secondly, we recall that the recursion operators are frequently nonlocal, thus requiring introduction of new nonlocal variables for the systems $\mathcal{E}$, see [5,6] for many examples. To this end, the systems are supplemented by differential relations (see (1) in section 1.3) which specify the new nonlocal dependent variables. Then these additional relations and their linearized counterparts are substituted but their symmetry flows are discarded, which is equivalent to setting them to zero. These substitution rules are included in `pdes` using `=>` in the form `df(f(i),x)=>`..., `d(1,f(i))=>`... and `df(b(j),x)=>`..., `d(1,b(j))=>`....

The order in which the `=` and `=>` relations occur in `pdes` is inessential.

Suppose the system in `pdes` is homogeneous w.r.t. several sets of homogeneity weights simultaneously, that is, it admits several scaling symmetries at once. One then wants to find a symmetry that obeys the same sets of homogeneity weights for the super-fields, each set supplemented by the expected weight of the parameter s. This is done as follows. An arbitrarily chosen (see a comment below) set of the weights `tw,sw` for the times t, s and `afwlist,abwlist` for the super-fields f_i, b_j is specified among the arguments of `ssym`, while the other weights are given through the global variable `hom_wei`. This is a list of lists `{sw_i,afwlist_i,abwlist_i}` with the additional sets of homogeneity weights of the parameter s and the dependent variables, respectively. The flag `filter` causes computation only of those symmetries that satisfy all additional homogeneities encoded in the global variable `hom_wei` as well.

The choice of the primary set of weights placed in the `ssym` arguments is arbitrary with the following comment: if all bosons' weights are positive and all fermions' weights are nonnegative in the primary set, then the bound `max_deg` is not applied even if `hom_wei` contains zero or negative weights of any variables. Indeed, in this case the primary weights already guarantee a finite number of terms in the ansatz for the symmetry, and the extra homogeneities in `hom_wei` act as an additional filter. The global bound `max_deg` is needed if for each set of homogeneity weights either one boson's weight is non-positive or one fermion's weight is negative.

Remark 2 The (non)uniqueness of the weights for a given system can be checked within some range by using the procedure `wgts`:

```
wgts(pdes,nf,nb,maxwt);
```

where

`pdes`	… list of equations `df(...,t) = ...`;
`nf`	… number of the fermion fields `f(1)`,…,`f(nf)`;
`nb`	… number of the boson fields `b(1)`,…,`b(nb)`;
`maxwt`	… only weights with a total sum $\leq$ `maxwt` are to be listed.

The weights are scaled such that the weight of D_x is always 1 and the weight of $\mathcal{D}$ is $\frac{1}{2}$. Note that the computer representation of the weights is twice the notation used throughout the paper to avoid half-integer values. This convention is usual.

Remark 3 The systems that admit several scaling symmetries and hence are homogeneous w.r.t. different weights allow to apply the breadth search method for recursions, which is the following. Let a recursion of weight $|s_R|$ w.r.t. a particular set of weights for the super-fields f, b and the time t be known. Now, recalculate its weight $|s'_R|$ w.r.t. another set of homogeneity weights $|f|$, $|b|$ and then find all recursion operators of weight $|s'_R|$. The list of solutions will incorporate the known recursion and, possibly, other operators. Generally, their weights will be different from the weight of the original recursion w.r.t. the initial set of weights. Hence we repeat the reasonings for each new operator and thus select the weights $|s_R|$ such that the recursions exist. This method is a serious instrument for checking consistency of calculations and elimination of errors since the existence of a solution is always guaranteed. We used it while testing the second mode of `ssym` in the SsTools package. An example of implementing it is given in the test output, where the recursion (10) for the multiple homogeneous $\alpha = 2$ system (9) is constructed.

1.3 The classification of systems and recursions

Let us introduce some notation which will simplify classification of recursion operators for the systems at hand. Assume $\mathcal{R}$ is a recursion for an equation and consider the symbol ${}^{\text{layers}}_{\text{ord}}\mathcal{R}^{\sharp}_{\text{weight}}$. The subscripts 'ord' and 'weight' denote the differential order and the weight of the recursion $\mathcal{R}$, respectively.

Recall further that the nonlocal variables are defined for $N = 1$ super-equations $\mathcal{E}$ by trivializing [5,6,9] conserved currents, which are of the form $D_t(\rho) + \mathcal{D}Q \doteq 0$; here the right-hand side vanishes by virtue ($\doteq$) of the system $\mathcal{E}$ and all possible differential consequences from it. The standard procedure [5] suggests that this relation is the compatibility condition for a new 'nonlocal' variable (the nonlocality), say w, whose derivatives are set to

$$\mathcal{D}w := \rho \text{ and } w_t := \mp Q, \tag{1}$$

where the sign corresponds to the parity preserving ('$-$') or reversing ('$+$') time t and $\bar{t}$, respectively. Each nonlocality thus makes the conserved current trivial; the new variables can be bosonic or fermionic. Hence, starting with an equation $\mathcal{E}$, one calculates several conserved currents for it and *trivializes* them by introducing a *layer* of nonlocalities whose derivatives are still local differential functions. This way the number of fields is increased and the system is extended by new substitution rules. Moreover, it may acquire new conserved currents that depend on the nonlocalities and thus specify the second layer of nonlocal variables with nonlocal derivatives. Clearly, the procedure is self-reproducing. An example of fixing the derivatives of a nonlocal variable is given in the text output. So, one keeps computing conserved currents and adding the layers of nonlinearities until an extended system $\tilde{\mathcal{E}}$ is achieved such that its linearization $\operatorname{Lin}\tilde{\mathcal{E}}$ has a (shadow of a, [5]) symmetry $\mathcal{R}$; this symmetry of $\operatorname{Lin}\tilde{\mathcal{E}}$ is precisely the resursion for the extended system $\tilde{\mathcal{E}}$.

The superscript 'layers' (if non-empty) in the symbol ${}^{\text{layers}}_{\text{ord}}\mathcal{R}^{\sharp}_{\text{weight}}$ indicates the required number of layers of the nonlocal variables assigned to conserved currents. The symbol '$\sharp$' denotes the number of recursions found for a given differential order, weight, and given nonlocalities. In the sequel, we denote local recursion operators by L and nonlocal ones by N. The symbol Z is used to denote a nilpotent recursion operator whose powers are equal zero except for a finite set of them, and $\varSigma$ is a super-recursion that swaps the parities of the flows.

Table 1
The classification of coupled super-systems and their recursions with respect to the primary weights $|f| = |b| = |D_t| = |D_{\bar{t}}| = \frac{1}{2}$.

(2) $\begin{cases} f_t = \mathcal{D}b + fb, \\ b_t = \mathcal{D}f \end{cases}$	${}^{2}_{0}N^{1}_{-1\frac{1}{2}},\ {}^{2}_{\frac{1}{2}}N^{1}_{-2},\ {}^{2}_{\frac{1}{2}}N^{1}_{-2\frac{1}{2}},\ {}^{2}_{\frac{1}{2}}N^{1}_{-3}$
(4) $\begin{cases} f_t = \mathcal{D}b, \\ b_t = \mathcal{D}f + b^2 \end{cases}$	${}^{1}_{1}N^{1}_{-1}$
(9) $\begin{cases} f_t = -\alpha\, fb, \\ b_t = \mathcal{D}f + b^2 \end{cases}$	$\alpha = 2:\quad {}_{\frac{1}{2}}L^{1}_{-2},\ {}_{\frac{1}{2}}L^{1}_{-2\frac{1}{2}},\ {}^{0}_{0}Z^{1}_{-2},\ {}^{0}_{0}\varSigma^{1}_{-2},\ {}^{0}_{0}\varSigma^{1}_{-2\frac{1}{2}},\ {}^{0}_{0}\varSigma^{1}_{-2\frac{1}{2}};$ $\alpha = 1:\quad {}^{0}_{0}Z^{1}_{-2\frac{1}{2}},\ {}_{\frac{1}{2}}Z^{1}_{-3};\qquad \alpha = 4:\quad {}_{1}L^{1}_{-3\frac{1}{2}}$ $\beta := -1/\alpha:\quad \beta \in \left\{-\frac{3}{4}, -\frac{1}{3}, -\frac{1}{6}, -\frac{1}{8}, 1, \pm\frac{3}{2}, \pm 2, \frac{5}{2}, 3\right\}$
(11) $\begin{cases} f_{\bar{t}} = \mathcal{D}f + b^2, \\ b_{\bar{t}} = \mathcal{D}b + fb \end{cases}$	${}_{0}\varSigma^{1}_{-\frac{1}{2}},\ {}_{\frac{1}{2}}L^{1}_{-1},\ {}_{\frac{1}{2}}\varSigma^{1}_{-\frac{3}{2}}$

It turns out that the equations presented in Table 1 exhibit practically the whole variety of properties that super-PDE of mathematical physics possess. Let us discuss them in more detail.

2 A fermionic extension of the Burgers equation

The system
$$f_t = \mathcal{D}b + fb, \qquad b_t = \mathcal{D}f \tag{2}$$
is homogeneous w.r.t. a unique set of weights $|f| = |b| = \frac{1}{2}$, $|D_t| = \frac{1}{2}$, $|D_x| = 1$. System (2) admits symmetries (f_s, b_s) for all weights $|D_s| \geq \frac{1}{2}$.

We recognize that system (2) is related to the fermionic extension of the Burgers equation [6]. Namely, consider the fermionic super-field $\phi = \eta(t,x) + \theta\xi(t,x)$ of weight 0 whose derivatives are $\phi_t = f$, $\mathcal{D}\phi = b$ and which thus potentiates the second equation in (2). Then from the first equation in (2) we get the scalar equation $\phi_{tt} = \phi_x + \phi_t \cdot \mathcal{D}\phi$. Its component form is the evolutionary system

$$\eta_x = \eta_{tt} - \xi\eta_t, \quad \xi_x = \xi_{tt} - \xi\xi_t + \alpha \cdot \eta_t\eta_{tt}, \qquad \alpha = 1, \tag{3}$$

where the variable x is the evolution parameter (the time) and the coordinate t is the new spatial variable. System (3) is precisely the $\alpha = 1$ fermionic (super)extension of the Burgers equation upon the functions $b(t,x)$ and $w(t,x)$, see [6, Eq. (16)], rewritten by setting $\xi(t,x) = -b(x,t)$ and $\eta(t,x) = w(x,t)$. Hence we conclude that the scalar super-field equation $\phi_{tt} = \phi_x + \phi_t \cdot \mathcal{D}\phi$ provides a one-component representation for that system; we recall that a different one-component representation of the fermionic extension for the Burgers equation derived in [6] is an evolution equation on an associative algebra. The geometry of the fermionic (super-)extension (3) was extensively studied in [6]; in particular, recursion operators were constructed for its symmetries. We emphasize that the correlation between system (2) and the extended Burgers equation was not noticed in [18], where (2) was first described.

The geometry of system (2) itself is essentially nonlocal, which is illustrated by calculating the conserved currents using SsTools, trivializing them, and constructing the recursion operators that involve the new nonlocalities. It is very likely that system (2) has only one conserved current which is already in use for specifying ϕ. In the second layer, many nonlocal conservation laws and hence many new variables appear. This is discovered by SsTools as follows.

The calculation of conservation laws for evolutionary super-systems with homogeneous polynomial right-hand sides is performed by using the procedure `ssconl`:

```
ssconl(N,tw,mincw,maxcw,afwlist,abwlist,pdes);
```

where

`N`	... the number of superfields θ^i;
`tw`	... 2 times the weight $\lvert D_t \rvert$;

`mincw`	... minimal weight of the conservation law;
`maxcw`	... maximal weight of the conservation law;
`afwlist`	... list of weights of the fermionic fields `f(1)`,...,`f(nf)`;
`abwlist`	... list of weights of the bosonic fields `b(1)`,...,`b(nb)`;
`pdes`	... list of the equations for which a conservation law must be found.

The ansatz for the differential polynomial components of a conserved current is composed in full generality (certainly, it has nothing to do with the axioms for the symmetry flows). Again, the global positive integer variable `max_deg` determines the highest power of a bosonic variable of nonpositive weight or a fermionic variable of negative weight and all their (super-)derivatives in any ansatz. The procedure `ssconl` (and `wgts` and `linearize` as well) is indifferent w.r.t. the presence of assignments `=` and `=>` in `pdes`.

The fact that the current is conserved on a given system `pdes` leads to the algebraic system for the undetermined coefficients, which is further solved automatically by CRACK [16]. Having obtained a conserved current, we define the new bosonic or fermionic dependent variable (the nonlocality) using the standard rules (1). An example illustrating the run of `ssconl` is given in the test output.

So, in addition to the potential ϕ, let us construct the fermionic variable v whose weight $|v| = \frac{3}{2}$ is minimal (other admissible nonlocal variables have greater weights): we calculate conservation laws involving ϕ and then we set $v_t = \mathcal{D}b\cdot\phi f b + f_x\phi f$ and $\mathcal{D}v = -\mathcal{D}b\cdot fb + \mathcal{D}f\cdot\mathcal{D}b\cdot\phi + b_x\phi f$. Now there appear nontrivial solutions to the determining equations for recursion operators. By convention, the capital V denotes the 'linearized' counterpart of v and likewise Φ for ϕ. Recall that the correlation between the 'linearized' variables in the solutions $\mathcal{R}$ obtained by SsTOOLS and the recursion operators R that act on the symmetries is as follows, see (6): the local variables F, B, and their derivatives denote the corresponding components of the flows, and the linearized nonlocalities, e.g. Φ, act by the rules that follow from their derivation formulas; we thus have the interpretation $\Phi = \mathcal{D}^{-1}(B)$.

We obtain the recursion of zero differential order with nonlocal coefficients:

$$\mathcal{R}_{[-1\frac{1}{2}]} = \begin{pmatrix} -\mathcal{D}b\cdot\phi f B + \phi v F + v\cdot B \\ \mathcal{D}b\,\phi f F - vF + v\phi\cdot B \end{pmatrix}.$$

Also, we get another operator, which is nonlocal and has nonlocal coefficients,

$$\mathcal{R}_{[-2]} = \begin{pmatrix} \mathcal{D}b\,V\phi - \mathcal{D}f\,\mathcal{D}B\,\phi f - \mathcal{D}f\,\mathcal{D}b\,\Phi f + \mathcal{D}f\,\mathcal{D}b\,F\phi + \mathcal{D}f\,V + V\phi f b \\ \mathcal{D}B\,\mathcal{D}b\,\phi f + \mathcal{D}b\,V - \mathcal{D}b\,F\phi f b + \mathcal{D}f\,\mathcal{D}b\,V\phi f + \mathcal{D}f\,V\phi - V f b \end{pmatrix};$$

it contains ϕ as well as Φ and V. The coefficients of the recursions $\mathcal{R}_{[-5/2]}$ and $\mathcal{R}_{[-3]}$ found for $|s_R| = -2\frac{1}{2}$ and $|s_R| = -3$ are also nonlocal.

3 A super-field representation for the Burgers equation

Consider the system

$$f_t = \mathcal{D}b, \qquad b_t = \mathcal{D}f + b^2, \tag{4}$$

which has the unique set of weights $|f| = |b| = \frac{1}{2}$, $|D_t| = \frac{1}{2}$, $|D_x| = 1$. Equation (4) admits the infinite sequence (7) of higher symmetries $f_s = \phi^f$, $b_s = \phi^b$ at all (half-)integer weights $|D_s| \geq \frac{1}{2}$. Also, there is another infinite sequence (8) of symmetries for (4) at all (half-)integer weights $|D_{\bar{s}}| \geq \frac{1}{2}$ of the odd 'time' $\bar{s}$.

System (4) is obviously related to the bosonic super-field Burgers equation

$$b_x = b_{tt} - 2bb_t, \qquad b = b(x, t, \theta). \tag{5}$$

We emphasize that the role of the independent coordinates x and t is reversed w.r.t. the standard interpretation of t as the time and x as the spatial variable. The Cole–Hopf substitution $b = -u^{-1}u_t$ from the heat equation $u_x = u_{tt}$ provides the solution for the bosonic component of (4).

Let us introduce the bosonic nonlocality $w(x, t, \theta)$ of weight $|w| = 0$ by trivializing the conserved form of the first equation in (4). Namely, we specify the derivatives of w by setting

$$\mathcal{D}w = -f, \quad w_t = -b.$$

Note that the variable w is a potential for both fields f and b. The nonlocality w satisfies the potential Burgers equation $w_x = w_{tt} + w_t^2$ such that the formula $w = \ln u$ gives the solution; the relation $f = -\mathcal{D}w$ determines the fermionic component in system (4).

Now we extend the set of dependent variables f, b, and w by the symmetry generators F, B, and W that satisfy the respective linearized relations,

$$F_t = \mathcal{D}B, \quad B_t = \mathcal{D}F + 2bB, \quad \mathcal{D}W = -F, \quad W_t = -B.$$

The linearization correspondence between the fields is $f \mapsto F$, $b \mapsto B$, and $w \mapsto W$. In this setting, we obtain the recursion of weight $|s_R| = -1$:

$$\mathcal{R}_{[-1]} = \begin{pmatrix} F_x - \mathcal{D}f\, F + f_x\, W \\ B_x - \mathcal{D}f\, B + b_x\, W \end{pmatrix} \iff R = \begin{pmatrix} D_x - \mathcal{D}f + f_x\, \mathcal{D}^{-1} & 0 \\ b_x\, \mathcal{D}^{-1} & D_x - \mathcal{D}f \end{pmatrix}. \tag{6}$$

The method for constructing $\mathcal{R}_{[-1]}$ is illustrated in the test output.

The recursion $\mathcal{R}_{[-1]}$ provides two sequences of higher symmetries for system (4):

$$\begin{pmatrix} f_t \\ b_t \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{D}b_x - \mathcal{D}f\mathcal{D}b - f_x b \\ \mathcal{D}f_x - (\mathcal{D}f)^2 - b^2\mathcal{D}f + bb_x \end{pmatrix} \mapsto \cdots, \quad \begin{pmatrix} f_x \\ b_x \end{pmatrix} \mapsto \begin{pmatrix} f_{xx} - 2\mathcal{D}f f_x \\ b_{xx} - 2\mathcal{D}f b_x \end{pmatrix} \mapsto \cdots. \tag{7}$$

The same recursion proliferates two experimentally found first order flows to two infinite sequences of symmetries with the odd parameters $\bar{s}$:

$$\begin{pmatrix} \mathcal{D}f \\ \mathcal{D}b \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{D}f_x - (\mathcal{D}f)^2 - f_x f \\ \mathcal{D}b_x - \mathcal{D}f\,\mathcal{D}b - b_x f \end{pmatrix} \mapsto \cdots, \quad \begin{pmatrix} f\mathcal{D}b - b\mathcal{D}f + b_x \\ b\mathcal{D}b - f\,\mathcal{D}f + f_x - fb^2 \end{pmatrix} \mapsto \cdots. \tag{8}$$

Remark 4 System (4) is not a supersymmetric extension of (5); it is a representation of the bosonic super-field Burgers equation. The flows in (7) become purely bosonic in the coordinates $\mathcal{D}f$, b. The standard recursion $R = D_t + \frac{1}{2}b + \frac{1}{2}b_t D_t^{-1}$ for the Burgers equation acts 'across' the two sequences in (7) and maps $(f_t, b_t) \mapsto (f_x, b_x)$; again, we note that the independent coordinates are reversed in (5).

However, from the above reasonings we gain two sequences of symmetries (8), which are *not* reduced to the bosonic (x,t)-independent symmetries of the Burgers equation. We finally recall that the Burgers equation (5) has infinitely many higher symmetries [19] that depend explicitly on the base coordinates x, t and thus exceed the set of axioms on p. 5.

4 A multiplet of super-systems

In this section we consider the systems

$$f_t = -\alpha f b, \qquad b_t = \mathcal{D}f + b^2. \tag{9}$$

Here the normal growth equation upon

$$f(t,x;\theta) = f(0,x;\theta) \cdot \exp\left(-\alpha \int_0^t b(\tau,x;\theta)\,\mathrm{d}\tau\right)$$

is coupled with the perturbed blow-up equation upon $b(t,x;\theta)$. System (9) has multiple homogeneities, and we let the tuple $|f| = |b| = \frac{1}{2}$, $|D_t| = \frac{1}{2}$, $|D_x| = 1$ be the primary 'reference system.' The symmetry integrable cases differ by the values of the parameter α. We discover that for $\alpha = 1$, 2, and 4 systems (9) admit the symmetries with both even and odd times s, $\bar{s}$. The equations in this triplet demonstrate different properties. The geometry of the $\alpha = 2$ system is quite extensive: this system admits a continuous sequence of symmetries for all (half-)integer weights $|D_s| \geq \frac{1}{2}$, four local recursions (one is nilpotent), and three local super-recursions. The equation for $\alpha = 1$ admits fewer structures, and the case $\alpha = 4$ for (9) is rather poor. All the three equations possess higher symmetries with parity reversing times $\bar{s}$. In what follows, we indicate the presence of recursion operators for these systems in certain weights $|s_R|$, $|\bar{s}_R|$.

Next, we find values of the parameter $\beta = -1/\alpha$ such that the respective systems (9) possess symmetries with parity reversing times $\bar{s}$ but have no higher commuting flows with the times s. Eleven cases are then realized and we specify the admissible weights $|\bar{s}|$ for each β.

4.1 Case $\alpha = 2$.

First we fix $\alpha = 2$ and consider (9): we get $f_t = -2fb$, $b_t = \mathcal{D}f + b^2$. The weights for symmetries are $|D_s| = \frac{1}{2}$, $|D_s| = 1$, and then equation (9) admits a continuous chain of symmetry flows for all (half-)integer weights $|D_s| \geq 2\frac{1}{2}$. Surprisingly, no nonlocalities are needed to construct the recursion operators, although there are many conservation laws for this system. Hence we obtain purely local recursion operators $\mathcal{R}$ that propagate the symmetries: $\varphi = (F, B) \mapsto \varphi' = \mathcal{R}$ for any φ. We get an operator of weight $-3\frac{1}{2}$,

$$\mathcal{R}_{[-7/2]} = \begin{pmatrix} 0 \\ (\mathcal{D}f)^3 fF + 6(\mathcal{D}f)^2 fb^2 F + 12\mathcal{D}f \cdot fb^4 F + 8fb^6 F \end{pmatrix}, \tag{10}$$

which is triangular since $\mathcal{R}^b$ does not contain B and, moreover, which is nilpotent. The above recursion is a recurrence relation [6] that is well-defined for all symmetries of (9). The recursion $\mathcal{R}_{[-2]}$ of weight $|s_R| = -2$ is also triangular; we have

$$\mathcal{R}_{[-2]} = \begin{pmatrix} \frac{11}{2}\mathcal{D}F\,\mathcal{D}f\,f + 11\mathcal{D}F\,fb^2 + \frac{3}{2}(\mathcal{D}f)^2 F + 3\mathcal{D}f\,Fb^2 + \frac{1}{2}f_x Ff \\ \\ 11\mathcal{D}B\,fb^2 + 8\mathcal{D}b\,Fb^2 + 22\mathcal{D}b\,fBb + 7(\mathcal{D}f)^2 B + \\ 14\mathcal{D}f\,Bb^2 + \frac{11}{2}\mathcal{D}f\,\mathcal{D}B\,f + \frac{5}{2}\mathcal{D}f\,\mathcal{D}b\,F + \frac{1}{2}b_x Ff + f_x Fb + 5f_x fB \end{pmatrix},$$

Further, we obtain the recursion $\mathcal{R}_{[-5/2]}$ of weight $2\frac{1}{2}$; its components are

$$\begin{aligned}
\mathcal{R}^f_{[-2\frac{1}{2}]} &= -2\mathcal{D}b\,Ffb^2 - \mathcal{D}F\,\mathcal{D}f\,fb - \mathcal{D}F\,fb^3 - \tfrac{1}{2}f_x Ffb - 2\mathcal{D}f\,fBb^2, \\
\mathcal{R}^b_{[-2\frac{1}{2}]} &= \mathcal{D}B\,fb^3 + \mathcal{D}b\,Fb^3 + \mathcal{D}b\,fBb^2 + \tfrac{1}{8}\mathcal{D}f_x Ff + \\
&\quad + \tfrac{1}{2}\mathcal{D}Fb^4 + \tfrac{1}{2}\mathcal{D}F(\mathcal{D}f)^2 + \mathcal{D}F\,\mathcal{D}f\,b^2 + \tfrac{1}{8}\mathcal{D}F\,f_x f + (\mathcal{D}f)^2 Bb + \\
&\quad + \mathcal{D}f\,Bb^3 + \mathcal{D}f\,\mathcal{D}B\,fb + \mathcal{D}f\,\mathcal{D}b\,Fb + \mathcal{D}f\,\mathcal{D}b\,fB + \tfrac{3}{8}\mathcal{D}f\,F_x f + \\
&\quad + \tfrac{1}{4}\mathcal{D}f\,f_x F + \tfrac{1}{2}b_x Ffb + \tfrac{1}{2}F_x fb^2 + \tfrac{1}{4}f_x Fb^2 + \tfrac{1}{2}f_x fBb.
\end{aligned}$$

The local recursion with $|s_R| = -3$ is huge.

For $\alpha = 2$, system (9) admits at least three super-recursions ${}^t(R^f,\, R^b)$ such that the parities of R^f and R^b are opposite to the odd parity for f (and hence for F) and to the even parity of b and B. This is possible owing to the presence of the variable $\bar{s}_R$. The triangular zero-order super-recursions are $\bar{\mathcal{R}}^f_{[-2]} = 4\mathcal{D}f\,Ffb + 8Ffb^3$, $\bar{\mathcal{R}}^b_{[-2]} = -4\mathcal{D}b\,Ffb + 2(\mathcal{D}f)^2F + 6\mathcal{D}f\,Fb^2 + 4\mathcal{D}f\,fBb\ - f_xFf + 4Fb^4 + 8fBb^3$ and

$$\bar{\mathcal{R}}_{[-2\frac{1}{2}]} = \begin{pmatrix} -\mathcal{D}f\,f_xF - 2f_xFb^2 \\ \mathcal{D}b\,f_xF - \mathcal{D}f\,b_xF + \mathcal{D}f\,f_xB - 2b_xFb^2 + 2f_xBb^2 \end{pmatrix}$$

for the weights $|\bar{s}_R| = -2$ and $|\bar{s}_R| = -2\frac{1}{2}$, respectively; the third super-recursion found for $|\bar{s}_R| = -2\frac{1}{2}$ is very large. Quite naturally, system (9) has infinitely many $\bar{s}$-symmetries if $\alpha = 2$.

4.2 Case $\alpha = 1$.

Setting $\alpha = 1$ in (9), we obtain the system $f_t = -fb$, $b_t = \mathcal{D}f + b^2$. The default set of weights is again $|f| = |b| = \frac{1}{2}$, $|D_t| = \frac{1}{2}$, and $|D_x| = 1$. The sequence of symmetries is not continuous and starts later than for the chain in the case $\alpha = 2$. We find out that there are symmetry flows if either $|D_s| = |D_t| = \frac{1}{2}$ (the equation itself), $|D_s| = |D_x| = 1$ (the translation along x), or $|D_s| \geq 3\frac{1}{2}$ such that a continuous chain starts for all (half-)integer weights $|D_s|$.

Similarly to the previous case, no nonlocalities are needed to construct the recursions, which therefore are purely local. The recursion operator $\mathcal{R}^f_{[-5/2]} = 0$, $\mathcal{R}^b_{[-5/2]} = (\mathcal{D}f)^2\,Ff + 3\mathcal{D}f\,Ffb^2 + \frac{9}{4}Ffb^4$ of maximal weight $|s_R| = -2\frac{1}{2}$ is nilpotent: $\mathcal{R}^2 = 0$. For the succeeding weight $|s_R| = -3$, we obtain a nilpotent local recursion $\mathcal{R}_{[-3]}$ whose components are given through

$$\begin{aligned}
\mathcal{R}^f_{[-3]} &= \tfrac{5}{3}\mathcal{D}F\,(\mathcal{D}f)^2f + \tfrac{5}{2}\mathcal{D}F\,\mathcal{D}f\,fb^2 - \tfrac{5}{3}(\mathcal{D}f)^3F - \tfrac{5}{2}(\mathcal{D}f)^2Fb^2 + \\
&\quad + 5\mathcal{D}f\,\mathcal{D}b\,Ffb + \tfrac{20}{3}\mathcal{D}f\,f_xFf + \tfrac{15}{2}f_xFfb^2, \\
\mathcal{R}^b_{[-3]} &= \mathcal{D}f_x\,Ffb - \tfrac{105}{2}\mathcal{D}F\,\mathcal{D}b\,fb^2 - \tfrac{160}{3}\mathcal{D}F\,\mathcal{D}f\,\mathcal{D}b\,f + 11\mathcal{D}F\,f_xfb + \\
&\quad + \tfrac{5}{3}(\mathcal{D}f)^2\mathcal{D}Bf + \tfrac{5}{3}(\mathcal{D}f)^2\mathcal{D}b\,F + \tfrac{5}{2}\mathcal{D}f\,\mathcal{D}B\,fb^2 + \tfrac{5}{2}\mathcal{D}f\,\mathcal{D}b\,Fb^2 - \\
&\quad - 55\mathcal{D}f\,\mathcal{D}b\,fBb + \tfrac{17}{3}\mathcal{D}f\,b_xFf + \mathcal{D}f\,f_xfB + \tfrac{23}{2}b_xFfb^2 + \tfrac{183}{2}f_xfBb^2.
\end{aligned}$$

The differential order of $\mathcal{R}_{[-3]}$ is positive.

4.3 Case $\alpha = 4$.

Finally, let $\alpha = 4$; then system (9) acquires the form $f_t = -4fb$, $b_t = b^2 + \mathcal{D}f$. Again, the primary set of weights is $|f| = |b| = \frac{1}{2}$, $|D_t| = \frac{1}{2}$, $|D_x| = 1$. System (9) admits the symmetries (f_s, b_s) of weights $|D_s| = \frac{1}{2}$, 1 or $|D_s| \geq 3\frac{1}{2}$ w.r.t. the primary set above. This situation coincides with the case $\alpha = 1$. Again, no nonlocalities are needed for constructing the recursion $\mathcal{R}_{[-7/2]}$ of weight $|s_R| = -3\frac{1}{2}$; this operator is relatively big:

$$\begin{aligned}
\mathcal{R}^f_{[-3\frac{1}{2}]} &= -12\mathcal{D}b\,Ffb^4 - \mathcal{D}F\,(\mathcal{D}f)^2fb - 4\mathcal{D}F\,\mathcal{D}f\,fb^3 - 3\mathcal{D}F\,fb^5 - \\
&- 4(\mathcal{D}f)^2fBb^2 - 4\mathcal{D}f\,\mathcal{D}b\,Ffb^2 - \tfrac{2}{3}\mathcal{D}f\,f_xFfb - 12\mathcal{D}f\,fBb^4 - 2f_xFfb^3, \\
\mathcal{R}^b_{[-3\frac{1}{2}]} &= 3\mathcal{D}Bfb^5 + 3\mathcal{D}b\,Fb^5 + 9\mathcal{D}b\,fBb^4 + \tfrac{1}{9}\mathcal{D}f_x\,\mathcal{D}f\,Ff - \tfrac{1}{3}\mathcal{D}f_x\,Ffb^2 + \\
&+ \mathcal{D}F\,\mathcal{D}b\,fb^3 + \tfrac{1}{4}\mathcal{D}F\,(\mathcal{D}f)^3 + \tfrac{5}{4}\mathcal{D}F\,(\mathcal{D}f)^2b^2 + \tfrac{7}{4}\mathcal{D}F\,\mathcal{D}f\,b^4 + \\
&+ \tfrac{5}{18}\mathcal{D}F\,\mathcal{D}f\,f_xf + \tfrac{1}{2}\mathcal{D}F\,f_xfb^2 + (\mathcal{D}f)^3Bb + 4(\mathcal{D}f)^2Bb^3 + \\
&+ (\mathcal{D}f)^2\mathcal{D}b\,Fb + (\mathcal{D}f)^2\mathcal{D}b\,fB + \tfrac{2}{9}(\mathcal{D}f)^2F_xf + \tfrac{1}{6}(\mathcal{D}f)^2f_xF + \\
&+ 4\mathcal{D}f\,\mathcal{D}B\,fb^3 + 4\mathcal{D}f\,\mathcal{D}b\,Fb^3 + 10\mathcal{D}f\,\mathcal{D}b\,fBb^2 + \mathcal{D}f\,F_xfb^2 + \\
&+ \tfrac{2}{3}\mathcal{D}f\,f_xFb^2 + \tfrac{5}{3}\mathcal{D}f\,f_xfBb + 2b_xFfb^3 + F_xfb^4 + \tfrac{1}{2}f_xFb^4 + f_xfBb^3 + \\
&+ \tfrac{3}{4}\mathcal{D}F\,b^6 + \mathcal{D}F\,\mathcal{D}f\,\mathcal{D}b\,fb + (\mathcal{D}f)^2\mathcal{D}B\,fb + 3\mathcal{D}f\,Bb^5 + \tfrac{2}{3}\mathcal{D}f\,b_xFfb.
\end{aligned}$$

No nilpotent recursion operators were found for system (9) if $\alpha = 4$.

Remark 5 We observe that an essential part of recursion operators for supersymmetric PDE are nilpotent. At present, it is not clear how the nilpotent recursion operators contribute to the integrability of supersymmetric systems and what invariants they describe or symptomize. We emphasize that this property does not always originate from the rule '$f \cdot f = 0$', but this is an immanent feature of the symmetry algebras.

Problem 6 Construct an equation $\mathcal{E}$ that admits nilpotent differential recursion operators $\{R_1, \ldots \mid R_i^{n_i} = 0\}$ which generate an infinite sequence of symmetries φ, $R_{i_1}(\varphi)$, $R_{i_2} \circ R_{i_1}(\varphi)$, $\ldots$ for $\mathcal{E}$. Here we assume that at least two operators (without loss of generality, R_1 and R_2) do not commute and hence the flows never become zero.

4.4 The systems with parity reversing parameters $\bar{s}$

Now we consider eleven systems (9) that admit symmetries with parity reversing parameters $\bar{s}$ but do not possess any commuting s-flows except for the cases $s = t$, $|D_t| = \frac{1}{2}$, and $s = x$, $|D_x| \equiv 1$. These systems are enumerated by the parameter $\beta = -1/\alpha$ whose usage makes the ordering simpler: we conjecture that the systems possessing higher $\bar{s}$-symmetries constitute an infinite family corresponding, in particular, to (half-)integer positive β's. In Table 2 we show the numbers of symmetries with a certain weight $|\bar{s}|$ for each of the systems (9) specified by β. For convenience, we use the computer notation for the weights, that is, we multiply them by 2. Empty boxes correspond to no symmetries in that particular weight.

Table 2
The symmetry structure for special systems (9).

$\beta\downarrow$ | $2\lvert D_{\bar{s}}\rvert \rightarrow$	2	3	4	5	6	7	8	9	10
1	•	•							
3/2			•				•		
2					•				•
−3/2							•	• • •	• • •
−1/6							•	• •	•
5/2							•		
−2									•
−3/4									•
−1/3									•
−1/8									•
3									•

5 A system with parity reversing times

Let the time $\bar{t}$ and the parameters $\bar{s}$ be parity reversing. Then there is a unique system that satisfies the axioms and the weight assumptions $|f| = |b| = |D_{\bar{t}}| = \frac{1}{2}$,

$$f_{\bar{t}} = \mathcal{D}f + b^2, \quad b_{\bar{t}} = \mathcal{D}b + fb. \tag{11}$$

Clearly, the weights are uniquely defined in (11). The numbers of symmetries for system (11) are arranged in Table 3; note that the integer weights correspond to $|s|$ and the half-integer values stand for $|\bar{s}|$.

Table 3
The number of symmetries for system (11).

$\lvert D_s\rvert \in \mathbb{Z},\ \lvert D_{\bar{s}}\rvert \in \mathbb{Z}+\frac{1}{2}$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$
$\sharp$ sym	2	4	2	1	1

The flows commuting with (11) are proliferated by three local recursion operators. The nilpotent super-recursion $\mathcal{R}_{[-1/2]}$ of weight $|\bar{s}_R| = -\frac{1}{2}$ and the local recursion with $|s_R| = -1$ are given through

$$\mathcal{R}_{[-\frac{1}{2}]} = \begin{pmatrix} fF \\ \frac{1}{2}bF - fB \end{pmatrix}, \qquad \mathcal{R}_{[-1]} = \begin{pmatrix} b \cdot \mathcal{D}B - \mathcal{D}b \cdot B - \mathcal{D}f \cdot F \\ \frac{1}{2}b \cdot \mathcal{D}F - \mathcal{D}f \cdot B \end{pmatrix}. \tag{12}$$

Another parity-reversing recursion for system (11) is

$$\mathcal{R}_{[-1\frac{1}{2}]} = \begin{pmatrix} f\mathcal{D}b \cdot B + f\mathcal{D}f \cdot F - fb \cdot \mathcal{D}B \\ \frac{1}{2}b\mathcal{D}b \cdot B + \frac{1}{2}bf \cdot \mathcal{D}F - \frac{1}{2}b^2 \cdot \mathcal{D}B + \frac{1}{2}b\mathcal{D}f \cdot F - f\mathcal{D}f \cdot B \end{pmatrix},$$

here we have $|\bar{s}_R| = -1\frac{1}{2}$.

In the test output, we present the SsTools calls that demonstrate how the recursion $\mathcal{R}_{[-1/2]}$ is obtained and we explain how the output is translated into the geometric language.

6 Conclusion

Using the presented computational techniques, we discovered that four classes of nonlinear coupled boson-fermion systems satisfy the set of axioms, which are believed natural in view of the nontriviality and nondegeneracy they ensure. Two of the classes are related to the Burgers equation, which is one of the most relevant and model-like equations of mathematical physics. Namely, they are a fermionic (super-)extension and a supersymmetric representation of the Burgers equation, respectively. The latter admits symmetries such that both the bosonic and fermionic fields are involved and such that the flows do not retract to the standard, purely commutative setting. Also, we found that a unique evolutionary system in this set of homogeneity weights has the parity reversing time $\bar{t}$. Finally, we obtained a multiplet of coupled normal evolution and explosion systems; the multiplet seems to be essentially infinite containing all (half)integer values $\beta \in \frac{1}{2}\mathbb{N}$ of the parameter at the least. Only three of the systems from this multiplet corresponding to $\alpha \in \{1, 2, 4\}$ admit higher symmetry flows with the parity preserving parameters s; the parameters $\bar{s}$ for higher symmetries of other systems are parity reversing. We do not know whether these three systems are distinguished by any relation to the three classical sequences of the complex semi-simple Lie algebras. Hence we see that the suggested classification scheme yields both generalizations of known integrable systems and also new models, which are expected to be relevant in nature in view of their vast symmetry properties: recursion operators were constructed for the above systems.

The search for super-systems together with their symmetries and further revealing the recursion operators were performed by the SsTools package, which was used in two modes, respectively. The breadth first search method for the recursions, see Remark 3, which is valid for the systems that admit multiple homogeneities, proved to be a serious help in debugging SsTools. Thus we conclude that the general classification problem (see section 1.1), which is an immense task of modern computer physics, is becoming tractable with this computer algebra package.

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TEST RUN OUTPUT

Here we illustrate the process of computing symmetries, recursions, and conserved currents by SSTOOLS. First we consider the weights $|f| = |b| = |D_t| = \frac{1}{2}$

and find all systems that satisfy these homogeneity weights and admit higher symmetries. We discover that system (11) is a unique solution obtained by the call

```
ssym(1,1,sw,{1},{1},{},{},{},{tpar,spar});
```

where `sw` $\in \{1,2,3,4,5\}$ and the flag `spar` is omitted whenever `sw` is even. The weights for this system are uniquely defined, which is confirmed by the call

```
wgts({df(f(1),t)=b(1)**2+d(1,f(1)),
      df(b(1),t)=f(1)*b(1)+d(1,b(1))},{1},{1},maxwt);
```

here `maxwt` can be any positive integer.

Now we present the intermediate computations that result in the recursion $\mathcal{R}_{[-1/2]}$ for (11), see (12). We compute the linearization of system (11), see section 1.2 for its definition:

```
linearize({df(f(1),t)=b(1)**2+d(1,f(1)),
           df(b(1),t)=f(1)*b(1)+d(1,b(1))},1,1);
```

The output contains system (11) and its linearization:

```
df(f(2),t)=2*b(2)*b(1) + d(1,f(2));
df(b(2),t)=d(1,b(2)) + f(2)*b(1) + f(1)*b(2);
```

The linearization correspondence between the fields is $f = \mathtt{f}(1) \mapsto F = \mathtt{f}(2)$ and $b = \mathtt{b}(1) \mapsto B = \mathtt{b}(2)$; the scaling weights of $F =$ `f(2)` and $B =$ `b(2)` are equal to the weights of $f =$ `f(1)` and $b =$ `b(1)`. Note that the numbers `nf` and `nb` of fields are automatically doubled. The linearized system is calculated at most once in any run of SsTools; the output is pasted into the next call as is.

Now is the time to find the recursion such that $|\bar{s}_R| = -\frac{1}{2}$ and hence `sw`$(= -2|\bar{s}_R|) = 1$. The reason why the substitutions `=>` are used in the original equations is explained in Remark 1 on p. 9: by construction, the recursions are symmetries of the linearized system, and the original equations are used for intermediate substitutions only. We thus have

```
ssym(1,1,1,{1,1},{1,1},
  {df(f(1),t)=>b(1)**2+d(1,f(1)),
   df(b(1),t)=>f(1)*b(1)+d(1,b(1)),
   df(f(2),t)=2*b(2)*b(1) + d(1,f(2)),
   df(b(2),t)=d(1,b(2)) + f(2)*b(1) + f(1)*b(2)},
  {},{},{tpar,spar,lin});
```

The output contains a unique solution:

```
1 solution was found.
  df(f(2),s)=f(2)*f(1);
  df(b(2),s)= - 1/2*f(2)*b(1) + f(1)*b(2);
```

This is precisely the first recursion in (12) for system (11). The result is achieved by the intermediate call of CRACK [16] from `ssym`; the solver can be handled in full automatic mode by typing `g 1000`.

Let us present a similar calculation that yields the local recursion (10) for system (9) with $\alpha = 2$; note that this time the weights are not uniquely defined and we thus have the multiple homogeneity case, see Remark 3. The input is

```
hom_wei:={{16,{3,3},{2,2}}};
ssym(1,1,7,{1,1},{1,1},
  {df(f(1),t)=>-2*f(1)*b(1),
   df(b(1),t)=>d(1,f(1))+b(1)**2,
   df(f(2),t)=-2*f(2)*b(1)-2*f(1)*b(2),
   df(b(2),t)=d(1,f(2))+2*b(1)*b(2)},
  {},{},{lin,filter});
```

the output is the recursion (10):

```
1 solution was found.
  df(f(2),s)=0;
  df(b(2),s)=d(1,f(1))**3*f(2)*f(1)
    + 6*d(1,f(1))**2*f(2)*f(1)*b(1)**2
    + 12*d(1,f(1))*f(2)*f(1)*b(1)**4 + 8*f(2)*f(1)*b(1)**6;
```

Note that the admissible set of weights $|b| = |B| = 0$, $|f| = |F| = -\frac{1}{2}$ for system (9) and the value $|s''_R| = 1$ would require the upper bound `max_deg:=6;` for this solution.

By shifting the primary weight `sw` and/or the additional values `sw_i` in `hom_wei`, one can inspect how new recursion operators appear and vanish.

A computation of a conserved current using `ssconl` is done as follows; here we get the conserved current for system (2) having already extended it by the variable $\phi =$ `f(2)`, see p. 13. The input is

```
ssconl(1,1,5,5,{1,0},{1},
  {df(f(1),t)=d(1,b(1))+f(1)*b(1),
   df(b(1),t)=d(1,f(1)),
   df(f(2),t)=>f(1),
```

```
    d(1,f(2))=>b(1)});
```

Then the output produced by SsTools is

```
NEXT: BOSONIC CONSERVATION LAWS OF WEIGHT 5
>>>>> Non-trivial conservation law:
Pt = Db*f(1)*b - Df(1)*Db*f(2) - b *f(2)*f(1)
                                  x

Pd(1) =  Db*f(2)*f(1)*b + f(1) *f(2)*f(1)
                              x
```

By trivializing the conservation relation $D_t(\texttt{Pt}) + \mathcal{D}(\texttt{Pd(1)}) \doteq 0$, we construct the fermionic variable v of weight $|v| = \frac{3}{2}$, see section 2.