A Linear Sparse Systems Solver LSSS applied to the Classification of Integrable non-abelian Laurent ODEs

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Abstract

On this poster we characterize the class of sparse linear algebraic "selective" systems, list methods to solve them efficiently, report on the ongoing development of the corresponding computer algebra program LSSS *and describe an application that lead to its creation.*

1. Motivation

Sparse linear algebraic systems are well known to arise in the discretization of partial differential equations (PDEs) but there is a source of such systems of very different nature: systems arising from integrability investigations of differential equations. The computer algebra program LSSS aims at the symbolic solution of extremely large systems of the second type, i.e. of "selective" systems in contrast to "numerical" systems of the first type. The following table highlights the different nature of both types of sparse systems and indicates that selective systems may allow special solution techniques to be much more effective than known solution techniques for "numerical" systems:

type	"numerical" systems	"selective" systems
examples	systems resulting from a discretization of PDEs	systems resulting from a symmetry investigation of PDEs
value of free parameters when applying the solution of the linear system	any floating point numbers (boundary values of PDE)	0 or 1 (to isolate the individual symmetries)
number of zero-valued variables in solution	essentially none	most variables
initial sparsity	yes	yes
sparsity throughout the process of exact solution	no	yes
systems are overdetermined	no	yes
iteration schemes for large problems of that type are	useful	not useful
selective solution of short (eg. 1-term) equations is	not useful	useful

Table 1. Characterization of two different types of sparse linear systems

In contrast to numerical systems where, for example, the solution of a system from discretizing the heat equation does depend on the temperature at all points of the boundary, i.e. the system *can not* stay sparse for the whole exact solution process, selective systems are very different because they determine a finite number of discrete mathematical objects if they exist. An example of such a system is shown below to result in Lie-symmetry investigations of differential equations.

2. About LSSS the Linear Sparse Systems Solver

The special nature of selective systems allows a dedicated computer program LSSS (Linear Sparse Systems Solver) [1] running under the computer algebra system REDUCE [7] to be much more efficient than conventional computer programs for solving sparse systems.
Efficiency increasing measures are:

- Because of the existence of many variables that take the value of zero in the solution the systems involve 1-term equations which are utilized first to simplify the remaining system and generate more 1-term equations.
- The simplification of a system due to the vanishing of variables can be accomplished much faster than the simplification due to other substitutions.
- From the mathematical problem it is clear whether the type of a system is numerical or selective and thus to apply the most suitable technique from the start.
- Selective linear systems are typically formulated by separation of larger expressions. The complete separation and up-front formulation of the whole linear system can be avoided through a repeatedly selective splitting and thus formulation and solution of 1-term equations.
- Increasing the complexity of the mathematical problem (e.g. by a increased degree of the ansatz for symmetries or first integrals) the overdetermination and sparseness increases compensating partially the exploding size of the initial linear system if 1-term equations are used rigorously.

The development of LSSS is ongoing. In June 2013 the solver was enhanced to deal with linear systems that involve non-linearly occurring symbolic parameters at basically no loss of speed.

3. The Application Leading to the Development of LSSS

One step in demonstrating integrability of differential equations is to show the existence of ∞ many infinitesimal symmetries. For the Kontsevich system ([3])

$$u_t = uv + puv^{-1} - qv^{-1}, \qquad v_t = -vu + rvu^{-1} + qu^{-1} \tag{1}$$

where $u(t), v(t)$ are non-commutative variables (in particular, square matrices of arbitrary size) and p, q, r are commuting constant parameters, a symmetry

$$u_\tau = Q_1(u, v, u^{-1}, v^{-1}), \qquad v_\tau = Q_2(u, v, u^{-1}, v^{-1}) \tag{2}$$

is given through its generator $(Q_1, Q_2)^T$. A (non-commuting) polynomial ansatz for Q_i in u, v, u^{-1}, v^{-1} with undetermined coefficients a_i inserted into the symmetry conditions

$$D_\tau D_t\, u = D_t D_\tau\, u, \qquad D_\tau D_t\, v = D_t D_\tau\, v. \tag{3}$$

after splitting wrt. u, v, u^{-1}, v^{-1} results in an overdetermined linear system for a_i with parameters p, q, r.
Other applications of LSSS and of the package CRACK in this context include:

- computing full and trace first integrals ($\to$ large sparse linear systems),
- searching for a Lax-pair ($\to$ bi-linear inhomogeneous systems),
- identifying L, A pairs through conjugation ($\to$ many medium linear systems),
- computing a pre-Hamiltonian structure ($\to$ small linear systems),
- computation of a recursion operator ($\to$ medium linear systems),
- application of a recursion operator ($\to$ medium linear systems),
- classification of systems

$$u_t = uv + P(u, v, u^{-1}, v^{-1}), \qquad v_t = -vu + Q(u, v, u^{-1}, v^{-1}) \tag{4}$$

having symmetries ($\to$ bi-linear inhomogeneous systems)

Our investigation of system (1) makes it the first non-abelian system with non-polynomial right hand sides for which integrability could be shown, in this case by computing a Lax pair with spectral parameter [2] and the other steps above. With the program LSSS it was possible to compute Lie-symmetries of degree up to 16. The complete linear system that had been solved for that degree includes over 10^9 equations for 172 Mio variables. Despite of the majority of them being zero, the general solution is not trivial as it has 32 free parameters and its formulation in ASCII form requires already several mega byte.

To go that high in the degree had the advantage of veryfying that the found Lax pair generates all trace first integrals, because they are mapped to all existing symmetries thriough a pre-Hamiltonian operator that was found as well.

N	Maple 14				LinBox				LSSS
	solve		total		default		sparse		(Reduce)
	Sym+FI	Sym	Sym+FI	Sym	Sym+FI	Sym	Sym+FI	Sym	Sym+FI
3	.03	.03	.06	.06					.00
4	.09	.10	.19	.22		.02	.02	.02	.01
5	.30	.31	.63	.70			.12	.13	.01
6	1.07	.96	7.81	2.04		30.6	1.1	.90	.07
7	6.01	6.88	11.02	13.81			14.9	12.9	.16
8	21.55	17.23	47.28	35.65		3080	283.5	210	.63
9	78.34	69.08	154.40	131.90			2318	1812	2.10
10	312.3	273.1	587.6	508.4			21610	21210	7.22
11	1237	1127	2262	2015					26.67
12									91.27
13									402.70

Table 2. Run times in sec for MAPLE 14 *to solve the linear system (solve), to solve the system and substitute the solution in the symmetry generators (total), for* LINBOX *to solve the system with its default method (default) or its program for sparse systems (sparse) and for* LSSS *to solve the system* and *substitute the solution in the symmetry generators where N is the degree of the symmetry ansatz and Sym refers to the symmetry conditions and Sym+FI refers to the system of symmetry conditions and the necessary conditions $D_\tau FI = 0$ for the full first integral $FI = uvu^{-1}v^{-1}$.*

4. Current Work

Eight sytems of the form (4) that have at least one symmetry contain up to 5 constant parameters. Current work tries to find the least restrictions on the parameter values for which integrability of these eight families can be shown and thus first integrals, symmetries, a Lax Pair, a pre-Hamiltonian operator and a recursion operator can be determined.

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