Classification of 3-dimensional integrable scalar discrete equations *

S.P. Tsarev†
Institut für Mathematik
Technische Universität Berlin
Berlin, Germany
(tsarev@math.tu-berlin.de sptsarev@mail.ru)

T. Wolf
Department of Mathematics, Brock University
500 Glenridge Avenue, St.Catharines,
Ontario, Canada L2S 3A1
(twolf@brocku.ca)

Abstract. We classify all integrable 3-dimensional scalar discrete affine linear equations $Q_3 = 0$ on an elementary cubic cell of the lattice $\mathbb{Z}^3$. An equation $Q_3 = 0$ is called integrable if it may be consistently imposed on all 3-dimensional elementary faces of the lattice $\mathbb{Z}^4$. Under the natural requirement of invariance of the equation under the action of the complete group of symmetries of the cube we prove that the only nontrivial (non-linearizable) integrable equation from this class is the well-known dBKP-system.

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1. Introduction

Discrete differential geometry has been actively studied in the last decade (see e.g. [3, 4]) and has provided much insight into structures that are fundamental both to classical differential geometry and to the theory of integrable PDEs. In addition to such purely mathematical fields, results in discrete differential geometry have a great potential in computer graphics and architectural design (see [9]).

In what follows we consider the elementary combinatorial cubic cell K_n with the vertices $\{(i_1, \ldots, i_n) | i_s \in \{0, 1\}\}$. The field variables $f_{i_1 \ldots i_n} \in \mathbb{C}$, $i_s \in \{0, 1\}$, are associated to its 2^n vertices. We will use the

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short notation $\mathbf{f}$ for the set $(f_{00\ldots0}, \ldots, f_{11\ldots1})$ of all these 2^n variables (see Figures 1A, 1B).

An n-dimensional discrete system of the type considered here is given by an equation of the form

$$Q_n(\mathbf{f}) = 0, \tag{1}$$

on the field variables on the elementary cubic cell K_n. This equation may be extended to the other elementary cubic cells of the cubic lattice $\mathbb{Z}^n$ with vertices at integer points $(i_1, \ldots, i_n)$, $i_s \in \mathbb{Z}$, $n \geq 2$, in the n-dimensional space $\mathbb{R}^n$: with each vertex with integer coordinates $(i_1, \ldots, i_n)$, $i_s \in \mathbb{Z}$, we associate a scalar field variable $f_{i_1\ldots i_n} \in \mathbb{C}$ and assume that the equation (1) for every elementary cubic cell of $\mathbb{Z}^n$ is the same, after shifting the indices of $\mathbf{f}$ suitably.

In the last two decades the study of special classes of (1) which are "integrable" (in one sense or another) has become very popular. We give below only a brief account of the current state of this field of research, for a more detailed account cf. [1]–[4] and the references given therein. In fact, discrete integrable systems underlie many classical integrable nonlinear PDEs, like the Krichever-Novikov equation and other examples, the latter appear as a continuous limit along some of the discrete directions. Other well known classes of integrable geometric objects (with $n = 2$ and $n = 3$), like minimal surfaces, conjugate nets, constant curvature surfaces, Moutard nets, isothermic surfaces, orthogonal curvilinear coordinates etc., are also obtained as some smooth limits along some of the directions of the respective discrete system. The remaining discrete directions automatically provide us with a transformation known in the classical continuous geometric context as Jonas/Ribaucour/Bäcklund transformation between surfaces of the given class. On the other hand, starting from the classical non-linear superposition principles for the aforementioned transformations one can obtain precisely the underlying discrete system. One of the cornerstones of discrete differential geometry (the idea to look for cubic nonlinear superposition formulas of Bäcklund transformations of nonlinear integrable PDEs) was laid down in [5]. The duality between the smooth objects in any of the geometric classes of integrable smooth surfaces mentioned above and their Bäcklund-type transformations is therefore put into a symmetric form of a single discrete n-dimensional system and is encoded as the notion of $(n+1)$-*dimensional consistency* [3]:

An n-dimensional discrete equation (1) is called consistent, if it may be imposed in a consistent way on all n-dimensional faces of a $(n+1)$-dimensional cube.

This can also be understood as the possibility to take $\mathbb{Z}^{n+1}$ and prescribe the n-dimensional equation (1) to hold on every n-dimensional face of every elementary $(n+1)$-dimensional cube (of size 1, with edges parallel to the coordinate axes) *without* side relations to appear. For this reason the discrete equation (1) is often called a “face formula”. A precise definition of consistency, suitable for the class of discrete equations treated in this paper, will be formulated in the next section.

This paper is devoted to application of computer algebra systems REDUCE and FORM ([10]), in particular the REDUCE package CRACK ([11, 12]), to the classification of 3-dimensional integrable discrete systems.

The paper is organized as follows. In Section 2 we give a brief description of the known results on 2-dimensional integrable scalar discrete equations of type (1) and the precise definition of $(n+1)$-dimensional consistency condition for such discrete n-dimensional systems.

Section 3 is devoted to the classification of symmetry types of affine linear equations (1) for dimensions $n = 2, 3, 4$.

In Section 4 we describe the results of our computations (Theorem 2): the only nontrivial (non-linearizable) integrable scalar affine linear $3d$-discrete equation invariant w.r.t. the complete group of symmetries of the cube is given by the formula (5) below.

The technical details of the computations can be found in [13].

2. The setup

The simplest but very important class of 2-dimensional integrable discrete equations was investigated in detail in [1, 2]. They have the form

$$Q(f_{00}, f_{10}, f_{01}, f_{11}) = 0, \tag{2}$$

where f_{ij} are scalar fields attached to the vertices of a square (see Fig. 1A) with two main requirements:

1) Affine linearity. (2) is affine linear w.r.t. every f_{ij}, i.e. Q has degree 1 in any of its four variables:

$$\begin{aligned} Q &= c_1(f_{10}, f_{01}, f_{11}) f_{00} + c_2(f_{10}, f_{01}, f_{11}) \\ &= c_3(f_{00}, f_{01}, f_{11}) f_{10} + c_4(f_{00}, f_{01}, f_{11}) = \dots \\ &= q_{1111} f_{00} f_{10} f_{01} f_{11} + q_{1110} f_{00} f_{10} f_{01} + q_{1101} f_{00} f_{10} f_{11} + \dots + q_{0000}. \end{aligned}$$

2) Symmetry. Equation (2) should be invariant w.r.t. the symmetry group of the square or its suitably chosen subgroup.

A few other requirements were given in [1], in particular the formula (2) involved *parameters* attached to the edges of the square.

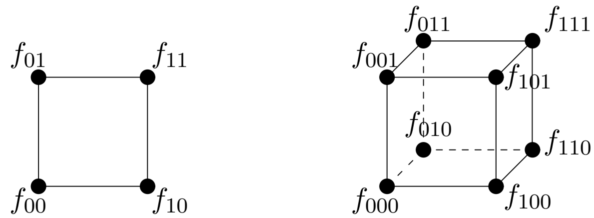

Figure 1. A. Square K_2. B. Cube K_3.

The second requirement of symmetry is obviously very important for the formulation of the condition of 3-dimensional consistency of (2). In fact, suppose we have a $3d$-cubic cell of $\mathbb{Z}^3$ (cf. Fig. 1B) and impose (2) to hold on three "initial $2d$-faces" $\{x_1=0\}$: $Q(f_{000}, f_{010}, f_{001}, f_{011})=0$; $\{x_2 = 0\}$: $Q(f_{000}, f_{100}, f_{001}, f_{101}) = 0$; $\{x_3 = 0\}$: $Q(f_{000}, f_{100}, f_{010}, f_{110}) = 0$ (these are used to find f_{011}, f_{101}, f_{110} from f_{000}, f_{100}, f_{010}, f_{001}). Then we impose (2) to hold on the other three "final $2d$-faces" $\{x_1 = 1\}$: $Q(f_{100}, f_{110}, f_{101}, f_{111}) = 0$; $\{x_2 = 1\}$: $Q(f_{010}, f_{110}, f_{011}, f_{111}) = 0$; $\{x_3 = 1\}$: $Q(f_{001}, f_{101}, f_{011}, f_{111}) = 0$; so for the last field variable f_{111} we can find 3 (a priori) different rational expressions in terms of the "initial data" f_{000}, f_{100}, f_{010}, f_{001}. The $3d$-*consistency* is the requirement that these three expressions of f_{111} in terms of the initial data should be identically equal. The subtle point of this process consists in the non-uniqueness of the mappings of a given square (Fig. 1A) onto the six $2d$-faces of the cube. The requirement of symmetry given above guarantees that we can choose any identification of the vertices of the 2-dimensional faces of the 3-dimensional elementary cube (Fig. 1B) with the vertices of the "standard" square where (2) is given; certainly this identification should preserve the combinatorial structure of the square (neighbouring vertices remain neighbouring). In [1] a complete classification of $3d$-consistent $2d$-discrete equations (in a slightly different setting) was obtained; in [2] a similar classification was given for the case when one does *not* assume that the formula (2) is the same on all the 6 faces of the 3-dimensional cube.

In the next sections we give a symmetry classification of all possible $3d$-discrete equations defined on some "standard" 3-dimensional cube:

$$Q(f_{000}, f_{100}, f_{010}, f_{001}, f_{110}, f_{101}, f_{011}, f_{111}) = 0 \qquad (3)$$

with respect to the complete symmetry group of the cube. Here, as everywhere in the paper, indices of the field variables f_{ijk} give the coordinates of the corresponding vertices of the standard $3d$-cube where our formula (3) is defined.

The requirement of consistency is now formulated similarly to the $2d$-case: given a $4d$-cube with field values f_{ijkl}, $i, j, k, l \in \{0, 1\}$, one should impose the formula (3) on every $3d$-face of it, by fixing one of the indices i, j, k, l, and making it 0 for the faces which we will call below "initial faces", or respectively 1 for the faces which we will call "final faces". One also needs to fix some mapping from the initial "standard" cube (with the vertices labelled f_{ijk}) onto every one of the eight $3d$-faces (for example $\{f_{i1kl}\}$ on $\{x_2 = 1\}$). This can certainly be done using the trivial lexicographic correspondence of the type $f_{ijk} \mapsto f_{i1jk}$. Geometrically this lexicographic correspondence is less natural since it is not invariant w.r.t. the symmetry group of a $4d$-cube. On the other hand there is an important example of such a non-symmetric formula corresponding to the discrete BKP equation ([1], equation (76)). Another possibility to avoid this problem is to impose the requirement of symmetry. More precisely, if one applies any one of the transformations from the group of symmetries of the $3d$-cube, (3) shall be transformed into an equation with the left hand side *proportional* to the original expression Q: $Q \mapsto \lambda \cdot Q$. Since this symmetry group is generated by reflections, one has $\lambda^2 = 1$, so this proportionality multiplier λ may be either $(+1)$ or (-1) for any particular transformation in the complete symmetry group.

From results in [1] we know that there are important $4d$-consistent $3d$-discrete equations which are preserved under a suitable *subgroup* of the complete symmetry group of the $3d$-cube. No classification of $3d$-discrete equations with such restricted symmetry property has beet carried out yet.

3. Symmetry classification

Every n-dimensional discrete equation $Q_n = 0$ which satisfies the requirement of affine linearity has a left hand side of the form

$$Q_n = \sum_{\mathcal{D}} q_{\mathcal{D}} \prod_{i_s=0,1} (f_{i_1\ldots i_n})^{D_{i_1\ldots i_n}} \tag{4}$$

with constant coefficients $q_{\mathcal{D}}$, where the summation is taken over all 2^{2^n} many 2^n-tuples $\mathcal{D} = (D_{00\ldots0}, \ldots, D_{11\ldots1})$, each power $D_{i_1\ldots i_n}$ of the respective vertex variable $f_{i_1\ldots i_n}$ being either 0 or 1. In other words: the 2^n indices of $q_{\mathcal{D}}$ are the exponents of the 2^n vertex field variables $f_{i_1\ldots i_n}$, each exponent $D_{i_1\ldots i_n}$ being 0 or 1. For example, $Q_2 = q_{1111} f_{00} f_{10} f_{01} f_{11} + q_{1110} f_{00} f_{10} f_{01} + q_{1101} f_{00} f_{10} f_{11} + \ldots + q_{0000}$ has $2^{2^2} = 16$ terms, Q_3 has respectively $2^{2^3} = 256$ terms, and Q_4 has already $2^{2^4} = 65536$ terms.

In this Section we classify n-dimensional affine linear equations $Q_n = 0$ for $n = 2, 3, 4$ that are invariant w.r.t. the complete symmetry group of the respective n-dimensional cube. This problem can be reduced to the enumeration of irreducible representations of this group in the space of polynomials of the form (4). Here this is done in a straightforward way: the group in question is generated by one reflection w.r.t the plane $x_1 = 1/2$ and $(n-1)$ diagonal reflections w.r.t. the planes $x_1 = x_s$, $s = 2, \dots, n$ (here x_k denote the coordinates in $\mathbb{R}^n$).

To every reflection R from this generating set we assign $(+)$ or $(-)$ and require the equality $Q(\mathbf{f}) = Q(R(\mathbf{f}))$ (respectively $Q(\mathbf{f}) = -Q(R(\mathbf{f}))$) to hold identically in all vertex variables $\mathbf{f}$; this gives us a set of equations for the coefficients $q_{\mathcal{D}}$. Running through all possible choices of the signs for the generating reflections we solve the united sets of simple linear equations for the coefficients $q_{\mathcal{D}}$ for every such choice. The main problem consists in the size of the resulting set of equations: for $n = 3$ we have for each combination of $\pm$ for the 3 generating reflections around 770 equation for the 256 coefficients $q_{D_1 \dots D_8}$; for $n = 4$ every set of equations for the coefficients of Q_4 has around 250,000 equations for the 65536 coefficients $q_{D_1 \dots D_{16}}$. Naturally, not every combination of signs for the generating reflections is possible, most of the resulting sets of equations for $q_{D_1 \dots D_{(2^n)}}$ allow only the trivial solution $q_{D_1 \dots D_{(2^n)}} = 0$. The results of our computations are formulated in Theorem 1:

THEOREM 1. *There are three nonempty symmetry classes of discrete equations Q_n for the dimensions $n = 2, 3$ and four nonempty classes for $n = 4$. The resulting number of free parameters in the coefficients $q_{\mathcal{D}}$ in each of the symmetric discrete equations Q_n (including an overall constant multiplier) and the number of nonzero terms are given for each of the nontrivial cases in Table I.*

In the notations of Table I for $n = 2$ the first sign in the second column refers to the reflection on the line $x_1 = 1/2$, the next sign stands for the reflection on the line $x_1 = x_2$. For example, the expressions of the first type $(+-)$ are invariant w.r.t. to the reflection on the line $x_1 = 1/2$, and show a change of sign after the reflection on the line $x_1 = x_2$. The last case $(++)$ consists of expressions which are invariant w.r.t. any element of the complete group of symmetries of the square which corresponds to the choice of the $(+)$ signs for the two generating reflections of the square w.r.t. the lines $x_1 = 1/2$ and $x_1 = x_2$.

Similarly to the case $n = 2$, there are only three cases $(+++)$, $(---)$ and $(-++)$ of nontrivial affine linear Q_3 for $n = 3$. The first sign here refers to a reflection on the plane $x_1 = 1/2$, the next signs stand for reflections on the planes $x_1 = x_2$, $x_1 = x_3$ (and $x_1 = x_4$ for $n = 4$).

Especially remarkable is the totally skew-symmetric case $(---)$ for $n = 3$: it has only one (up to a constant multiplier) nontrivial

Table I. Symmetry classification of the discrete equations w.r.t the complete symmetry group of the cube

n	types of symmetry, number of parameters and terms	number of parameters in SL_2-invariant subcases
2	1) $(+-)$: 1 param.; 4 terms 2) $(-+)$: 3 param.; 10 terms 3) $(++)$: 6 param.; 16 terms	1) 1 param.; 4 terms 2) none 3) 1 param.; 6 terms
3	1) $(---)$: 1 param.; 24 terms 2) $(-++)$: 13 param.; 186 terms 3) $(+++)$: 22 param.; 256 terms	1) 1 param.; 24 terms 2) none 3) 3 param.; 114 terms
4	1) $(----)$: 94 param.; 29208 terms 2) $(+---)$: 77 param.; 26112 terms 3) $(-+++)$: 349 param.; 60666 terms 4) $(++++)$: 402 param.; 2^{16} terms	1) 5 param.; 15480 terms 2) none 3) 3 param.; 15809 terms 4) 18 param.; 96314 terms

expression

$$Q_{(---)} = (f_{100} - f_{001})(f_{010} - f_{111})(f_{101} - f_{110})(f_{011} - f_{000}) - (f_{001} - f_{010})(f_{111} - f_{100})(f_{000} - f_{101})(f_{110} - f_{011}). \quad (5)$$

Precisely this expression gives the so called discrete Schwarzian Kadomtsev-Petviashvili (type B) system (dBKP-system) — an integrable discrete system found in [6, 7] and studied in [1] where the fact of its $4d$-consistency was first established. The dBKP-system has many equivalent forms and appears in very different contexts. In addition to the known geometric interpretations and a reformulation as Yang-Baxter system ([8]), the dBKP-system may be considered as a nonlinear superposition principle for the classical 2-dimensional Moutard transformations ([5]).

The expression (5) enjoys an extra symmetry property: the equation $Q_{(---)} = 0$ is invariant under the action of the $SL_2(\mathbb{C})$ group of fractional-linear transformations $\mathbf{f} \longmapsto (a\mathbf{f} + b)/(c\mathbf{f} + d)$ (all group parameters a, b, c, d are the same for all the vertices of the cube). This is in fact a direct consequence of the uniqueness of the integrable discrete equation (5) in this class. Since this $SL_2(\mathbb{C})$ action obviously preserves the symmetry type of an expression w.r.t. the group action of the cube symmetry group, it is reasonable to find the subclasses of

SL_2-invariant discrete equations in each symmetry class. This is given in the third column of Table I.

4. Nonexistence of nontrivial integrable discrete equations in other symmetry classes for $n = 3$

In this section we give a sketch of a computational proof of our main result:

THEOREM 2. *Among the three possible symmetry types of* 3*-dimensional affine linear discrete equations given in the second column of Table I, only formula (5) gives a non-trivial (i.e. non-linearizable)* 4*d-compatible discrete equation. Any* 4*d-compatible discrete equation in the other two symmetry types may be transformed using the action of the group* $SL_2(\mathbb{C})$ *on the field variables to one of the following linearizable forms:*

$$Q^{(1)} = f_{000}f_{001}f_{010}f_{011}f_{100}f_{101}f_{110}f_{111} - \sigma, \quad (6)$$

$$Q^{(2)} = f_{001}f_{010}f_{100}f_{111} - \sigma f_{000}f_{011}f_{101}f_{110}, \quad (7)$$

$$Q^{(3)} = (f_{001} + f_{010} + f_{100} + f_{111}) - \sigma(f_{000} + f_{011} + f_{101} + f_{110}), \quad (8)$$

where $\sigma = \pm 1$.

Technically, in order to find 4d-consistent 3d-discrete equations among the other two 3-dimensional cases $(-++)$ and $(+++)$ (as listed in Table I), one shall run the following algorithmic steps:

Step 1. Take a copy of this Q_3 formula, map it onto the four initial faces of the 4d-cube (where one of the coordinates $x_i = 0$), solve the mapped equations with respect to f_{0111}, f_{1011}, f_{1101} and f_{1110} leaving the other variables free.

Step 2. Then substitute the obtained *rational* expressions for f_{0111}, f_{1011}, f_{1101} and f_{1110} into the copies of the discrete equation mapped onto the four final faces (where one of the coordinates $x_i = 1$), finding respectively four different expressions for the last vertex field f_{1111}.

Step 3. Equate these 4 expressions for f_{1111} obtaining three rational equations in terms of 11 free variables f_{0000}, f_{0010}, f_{0101}, ... and the parametric coefficients $q_{D_1 \ldots D_8}$ left free in the chosen symmetry class.

Step 4. Removing the common denominators of the equations and splitting the resulting polynomials w.r.t. the 11 free variables f_{ijkl} one obtains a polynomial system of equations for the free coefficients $q_{D_1 \ldots D_8}$.

Step 5. The latter should be solved, resulting in a complete classification of 4d-consistent affine linear scalar 3d-discrete equations.

This approach, applied in a straightforward way, results in extremely huge expressions. Even building the rational expressions in Step 4 in a straightforward way seems to be unrealistic: for a typical $3d$-discrete equation from Table I Step 4 should end up (as our test runs allowed for an estimate) in an expression with around 10^{14} terms, which is beyond the reach of computer algebra systems in the foreseeable future. Even brute force verification of $4d$-consistency of the smallest solution (discrete equation (5) which has no free parametric coefficients $q_{D_1 \ldots D_8}$) results in $\approx 2 \cdot 10^8$ terms (after substituting the expressions for f_{0111}, f_{1011}, f_{1101}, f_{1110}, collecting the terms over the common denominator and expanding the brackets before the cancellation can start in Step 4, cf. [13]). Technically this is explained by the presence of 4 different symbolic denominators of the rational expressions for f_{0111}, f_{1011}, f_{1101}, f_{1110} and their various products. A careful step-by-step substitution and cancellation of like terms in several stages still can be done even on a currently standard PC *for this* $(- - -)$ *case*. Using Form (this symbolic computation system was specially designed for large computations), one can prove that all terms finally cancel out for the case of the integrable $3d$-discrete equation (5) thus giving a computational proof of its $4d$-consistency in 3 min CPU time (3 GHz Intel running Linux SUSE 9.3) and less than 200 Mb disk space for temporary data storage (cf. [13]).

As our preliminary runs had shown, the straightforward approach based on Steps 1–4 is unrealistic for the other two 3-dimensional cases listed in Table I, even when generating the consistency conditions in Step 4.

In order to classify discrete integrable $3d$-discrete equations $Q_3 = 0$ for the case $(- + +)$ and the hardest case $(+ + +)$ we used a totally different randomized "probing" strategy, explained in detail in [13].

After the computation (cf. [13] for the details) the list of candidate formulas Q_3 for the case $(+ + +)$ included 5 candidates (before the verification that these formulas, obtained by our "probing" method, really give $4d$-consistent $3d$-formulas). For the case $(- + +)$ the list of candidate formulas included 3 formulas. All of them include a few free parameters. As one can show, all these formulas can be greatly simplified using the action of the group $SL_2(\mathbb{C})$ on the field variables $\mathbf{f}$, resulting in the $4d$-consistent $3d$-discrete equations (6), (7), (8). The first two formulas (6), (7) can be easily linearized using the logarithmic substitution $\tilde{f}_{ijk} = \log f_{ijk}$.

The expressions for the aforementioned candidates, the Form procedures and their logfiles showing the simplification process can be downloaded from
`http://lie.math.brocku.ca/twolf/papers/TsWo2007/`.

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Address for Offprints: Prof. T. Wolf,
Department of Mathematics, Brock University
500 Glenridge Avenue, St.Catharines,
Ontario, Canada L2S 3A1
twolf@brocku.ca