SYMMETRIES AND CONSERVATION LAWS OF THE GENERALIZED KRICHEVER-NOVIKOV EQUATION

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Abstract. A computational classification of contact symmetries and higher-order local symmetries that do not commute with t, x, as well as local conserved densities that are not invariant under t, x is carried out for a generalized version of the Krichever-Novikov equation. Two main results are obtained. First, the Krichever-Novikov equation is explicitly shown to have a (new) local conserved density that involves t, x. Second, apart from the known dilational point symmetries for both the Krichever-Novikov equation and its generalized version, no other local symmetries with low differential order are found to involve t, x.

1. Introduction

The Krichever-Novikov (KN) equation was introduced in 1979 in the form [1]

$$c_t = \frac{3}{8}\frac{1 - c_{xx}^2}{c_x} - \frac{1}{2}Q(c)c_x^3 + \frac{1}{2}c_{xxx}$$

where $Q(c)$ is expressed in terms of the Weierstrass elliptic function (see also Ref.[2]). An alternative form of this equation is given by [3, 4, 5]

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{p(u)}{u_x} \tag{1}$$

where $p(u)$ is an arbitrary quartic polynomial

$$p(u) = C_1u^4 + C_2u^3 + C_3u^2 + C_4u + C_5 \tag{2}$$

with constant coefficients.

A main property of the KN equation, as shown in Refs.[3, 6, 7], is that it belongs to the class of integrable evolutionary equations in the sense of having an infinite number of higher-order local symmetries and higher-order local conserved densities, which are connected to Hamiltonian structures and recursion operators for the equation.

Among all of these equations, the KN equation is singled out by several features. For example, it is the only nonlinear integrable equation of the third order evolutionary form $u_t = u_{xxx} + F(u, u_x, u_{xx})$ that cannot be mapped to the Korteveg-de Vries (KdV) equation $v_t = v_{xxx} + vv_x$ by a finite-order differential substitution $v = \Phi(u, u_x, u_{xx}, \ldots)$. This result follows from the classifications of nonlinear integrable equations presented in Refs.[8, 9, 3, 6] and has been extended to a larger class of equations $u_t = a(u)u_{xxx} + F(u, u_x, u_{xx})$ in Ref.[5]. Also, it is the only nonlinear integrable equation in these classes that does not have a scaling symmetry. More generally, the KN equation (for an arbitrary quartic $p(u)$) has no known local symmetries that explicitly contain the variables t, x, and likewise none of its local conserved densities that are known to-date [6, 3, 10] have explicit dependence on t, x.

In this paper we address the question of whether the Krichever-Novikov equation admits any local symmetries or local conserved densities that have some essential dependence on t, x. We will in fact settle this question for a generalization of the Krichever-Novikov equation (called the *gKN equation*) given by

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x} \tag{3}$$

where $f(u) \not\equiv 0$ is a general differential function. This equation (3) is integrable only when $f(u)$ is a quartic polynomial (2), coinciding with the KN equation. The point symmetries of the gKN equation have been recently classified in Ref.[11, 12].

Symmetries that explicitly contain t, x are related to master symmetries [13], while conserved densities that explicitly contain t, x are typically important in the study of solitons and other exact solutions.

For the gKN equation (3), we will classify its admitted contact symmetries and higher-order local symmetries up to differential order six that do not commute with translations on t, x, as well as all of its admitted local conserved densities up to differential order three. Our results yield new conserved densities including one containing t, x that holds when the function $f(u)$ is an arbitrary quartic polynomial. Thus, we establish that the KN equation (1)–(2) does possess a local conserved density with essential dependence on t, x. We also rule out the existence of any local symmetries (with low differential order) that explicitly contain t, x, other than the known dilational point symmetries that hold only for certain (special) functions $f(u)$.

Our symmetry and conserved density classifications are derived in section 2 and section 3, respectively. In section 4, we work out the action of the point symmetries on the conserved densities. We make some concluding remarks in section 5.

From a computational viewpoint, the problem of classifying symmetries and conserved densities consists of solving a linear system of determining equations whose unknowns are functions of t, x, u, and derivatives of u (up to some finite differential order), together with the function $f(u)$. Most of the computation involves solving equations that are linear in the unknowns. After all dependencies of the unknowns on t, x and derivatives of u are determined from these equations, the remaining equations involve only the dependencies on u, including the function $f(u)$. The solution of these final equations is inherently a nonlinear problem, but the equations can be factorized such that the function $f(u)$ can be determined by solving an ODE (possibly nonlinear) while the dependencies of the unknowns on u can be determined by solving some linear equations. We carry out the computations by using the computer algebra programs LiePDE [14, 15] to compute symmetries and ConLaw [15, 16] to compute conservation laws. We have also verified the results by doing an interactive computation in Maple which also gave a more compact form for the solutions with fewer redundant special cases.

2. Classification of symmetries

To begin, consider the Lie symmetry group of the gKN equation (3). Since the gKN equation involves only a single dependent variable u, its Lie symmetry group comprises point symmetries and contact symmetries [17, 13, 18, 19].

A *point symmetry* of the gKN equation (3) is a group of transformations on (t, x, u) given by an infinitesimal generator

$$\mathrm{X} = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u \tag{4}$$

whose prolongation satisfies

$$\mathrm{prX}\Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big) = 0 \tag{5}$$

for all formal solutions $u(t, x)$ of the gKN equation (3). When acting on solutions, any point symmetry (4) is equivalent to an infinitesimal generator with the *characteristic form*

$$\hat{\mathrm{X}} = P\partial_u, \quad P = \eta(t, x, u) - \tau(t, x, u)u_t - \xi(t, x, u)u_x \tag{6}$$

where the characteristic functions η, τ, ξ are determined by

$$\begin{aligned} 0 &= \mathrm{pr}\hat{\mathrm{X}}\Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big) \\ &= D_tP - D_x^3P + 3\frac{u_{xx}}{u_x}D_x^2P - \frac{3}{2}\frac{u_{xx}^2}{u_x^2}D_xP + \frac{f(u)}{u_x^2}D_xP - \frac{f'(u)}{u_x}P \end{aligned} \tag{7}$$

holding for all formal solutions $u(t, x)$ of the gKN equation (3). This formulation is useful for doing computations and for considering extensions to contact symmetries and higher-order symmetries, as well as for making a connection with conserved densities.

A *contact symmetry* extends the definition of invariance (7) by allowing the transformations to depend essentially on first order derivatives of u, as given by an infinitesimal generator with characteristic form

$$\hat{\mathrm{X}} = P(t, x, u, u_t, u_x)\partial_u. \tag{8}$$

The corresponding transformations on (t, x, u, u_t, u_x) are given by

$$\mathrm{X} = \tau\partial_t + \xi\partial_x + \eta\partial_u + \eta^t\partial_{u_t} + \eta^x\partial_{u_x} \tag{9}$$

where

$$\tau = -P_{u_t}, \quad \xi = -P_{u_x}, \quad \eta = P - u_tP_{u_t} - u_xP_{u_x}, \quad \eta^t = P_t + u_tP_u, \quad \eta^x = P_x + u_xP_u \tag{10}$$

which follows from preservation of the contact condition $du = u_tdt + u_xdx$. Note that a contact symmetry reduces to a (prolonged) point symmetry if and only if P is a linear function of u_t and u_x.

The set of all infinitesimal point and contact symmetries admitted by the gKN equation (3) inherits the structure of a Lie algebra under commutation of the operators X. For a given (sub)algebra of point or contact symmetries, the corresponding group of transformations has a natural action [13, 18, 19] on the set of all solutions $u(t, x)$.

To classify all of the contact symmetries (and point symmetries) admitted by the gKN equation (3), we first substitute a general characteristic function $P(t, x, u, u_t, u_x)$ into the symmetry determining equation (7). Next we eliminate u_{xxx}, u_{txxx}, u_{xxxx} through writing the gKN equation in the solved form

$$u_{xxx} = u_t - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x} \tag{11}$$

and doing the same for its differential consequences. The determining equation (7) then splits with respect to u_{xx}, u_{tx}, u_{tt}, u_{txx} into a linear overdetermined system of equations on $P(t, x, u, u_t, u_x)$. We find that this system contains the equations

$$P_{u_t u_t} = 0, \quad P_{u_x u_x} = 0, \quad P_{u_t u_x} = 0, \tag{12}$$

which imply that P is linear in u_t and u_x. Hence P reduces to the characteristic form for a point symmetry (6). We then find that the remaining equations in the system are given by

$$\tau_u = 0, \quad \tau_x = 0, \quad \xi_u = 0, \quad \xi_{xxx} - \xi_t = 0, \quad \tau_t - 3\xi_x = 0, \tag{13}$$

$$\eta_t = 0, \quad \eta_x = 0, \quad \eta_{uuu} = 0, \tag{14}$$

$$2f(u)\,(\eta_u - 2\xi_x) - f'(u)\eta = 0. \tag{15}$$

These equations (13)–(15) are straightforward to solve. If $\eta = 0$, then we have

$$\tau = C_1, \quad \xi = C_2. \tag{16}$$

When $\eta \neq 0$, then instead we obtain

$$\tau = 3C_1 t + C_2, \quad \xi = C_1 x + C_3, \quad \eta = C_4 u^2 + C_5 u + C_6, \tag{17}$$

together with the condition

$$\frac{f'(u)}{f(u)} = \frac{2(2C_4 u + C_5 - 2C_1)}{C_4 u^2 + C_5 u + C_6}. \tag{18}$$

The integration of ODE (18) splits into five distinct cases: (i) $C_4 = C_5 = 0$; (ii) $C_4 = 0$, $C_5 \neq 0$; (iii) $C_4 \neq 0$, $C_5^2 - 4C_4C_6 = 0$; (iv) $C_4 \neq 0$, $C_5^2 - 4C_4C_6 > 0$; (v) $C_4 \neq 0$, $C_5^2 - 4C_4C_6 < 0$. This leads to the following classification result.

Theorem 2.1. *(i) For any $f(u)$, the gKN equation* (3) *admits no contact symmetries.*
(ii) The point symmetries admitted by the gKN equation (3) *for arbitrary $f(u)$ consist of*

$$\mathrm{X}_1 = \partial_t, \quad \mathrm{X}_2 = \partial_x. \tag{19}$$

(iii) The gKN equation (3) *admits additional point symmetries only for the following* $f(u) \not\equiv 0$*:*

$$\text{(a)} \qquad \begin{aligned} & f(u) = C\exp(4au), \quad a \neq 0 \\ & \mathrm{X}_{3\mathrm{a}} = -3at\partial_t - ax\partial_x + \partial_u \end{aligned} \tag{20}$$

$$\text{(b)} \qquad \begin{aligned} & f(u) = C(u+b)^{2-4a}, \quad a \neq -1/2 \\ & X_{3\mathrm{b}} = 3at\partial_t + ax\partial_x + (u+b)\partial_u \end{aligned} \tag{21}$$

$$\text{(c)} \qquad \begin{aligned} & f(u) = C(u+b)^4 \exp(4a/(u+b)), \quad a \neq 0 \\ & X_{3\mathrm{c}} = 3at\partial_t + ax\partial_x + (u+b)^2\partial_u \end{aligned} \tag{22}$$

$$\text{(d)} \qquad \begin{aligned} & f(u) = C(c+b+u)^{2+2a/c}(c-b-u)^{2-2a/c}, \quad c \neq 0, \quad a \neq \pm c \\ & X_{3\mathrm{d}} = 3at\partial_t + ax\partial_x + ((u+b)^2 - c^2)\partial_u \end{aligned} \tag{23}$$

$$\text{(e)} \qquad \begin{aligned} & f(u) = C((u+b)^2 + c^2)^2 \exp\left((4a/c)\arctan((u+b)/c)\right), \quad c \neq 0 \\ & X_{3\mathrm{e}} = -3at\partial_t - ax\partial_x + ((u+b)^2 + c^2)\partial_u \end{aligned} \tag{24}$$

$$\text{(f)} \qquad \begin{aligned} & f(u) = C(u+b)^4 \\ & X_{3\mathrm{f}} = -(3t/2)\partial_t - (x/2)\partial_x + (u+b)\partial_u = X_{3\mathrm{b}}|_{a=-1/2}, \\ & X_{4\mathrm{f}} = (u+b)^2\partial_u = X_{3\mathrm{c}}|_{a=0} \end{aligned} \tag{25}$$

Modulo equivalence transformations, this classification can be easily checked to reduce to the classification of point symmetries of the gKN equation stated in Ref.[11]. (The classification presented in Ref.[12] is missing cases (22) and (23).) The equivalence transformations of the gKN equation, other than symmetry transformations, consist of a scaling on t, x, u, and $f(u)$, as well as a Mobius transformation on u and $f(u)$ [11, 10]. Here, we are interested in getting an explicit classification without having to change the form of $f(u)$ under equivalence transformations.

A *higher-order symmetry* of the gKN equation (3) is an infinitesimal generator

$$\hat{\mathrm{X}} = P(t, x, u, u_x, u_{xx}, \ldots)\partial_u \tag{26}$$

whose characteristic function P has a differential order of at least two, satisfying the symmetry determining equation (7) for all formal solutions $u(t, x)$. Existence of higher-order symmetries that commute with translations on t, x is connected with integrability. Since the gKN equation is integrable only when it coincides with the KN equation (1)–(2), we expect that such symmetries $\hat{\mathrm{X}} = P(u, u_x, u_{xx}, \ldots)\partial_u$ will be admitted only when $f(u)$ is a quartic polynomial. This still leaves open the possibility for existence of higher-order symmetries that involve t, x explicitly, which would be related to a master symmetry.

To look for symmetries that involve t, x explicitly, we will now classify all higher-order symmetries (26) up differential order six admitted by the gKN equation (3). In particular, any such symmetries that are admitted for special $f(u)$ will be determined.

For computational purposes, it will be useful to use an equivalent representation for the symmetries (26), which is given by eliminating $u_{xxx}, \ldots, u_{xxxxxx}$ in P through the solved form of the gKN equation (11). Then we have the following correspondence result.

Lemma 2.1. *For the gKN equation* (3)*, symmetries up to differential order six in x derivatives*

$$\hat{\mathrm{X}} = P(t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx}, u_{xxxxxx})\partial_u \tag{27}$$

are equivalent to symmetries of the form

$$\hat{\mathrm{X}} = \widetilde{P}(t, x, u, u_t, u_x, u_{tx}, u_{xx}, u_{tt}, u_{txx})\partial_u \tag{28}$$

whose differential order in t, x derivatives is at most three (where $P_{u_{ttx}} = P_{u_{ttt}} = 0$).

The determining equation (7) for symmetries (28) splits with respect to u_{ttx}, u_{ttxx} into a linear overdetermined system of equations on

$$\widetilde{P}(t, x, u, u_t, u_x, u_{tx}, u_{xx}, u_{tt}, u_{txx}). \tag{29}$$

This system can solved in a straightforward computational way.

Theorem 2.2. *The gKN equation* (3) *admits a higher-order symmetry characteristic* (29) *only when $f(u) \not\equiv 0$ is a quartic polynomial:*

$$\begin{aligned} f(u) &= C_1u^4 + C_2u^3 + C_3u^2 + C_4u + C_5 \\ \widetilde{P} &= u_{txx} - 2\frac{u_{tx}u_{xx}}{u_x} + \frac{1}{2}\frac{u_t u_{xx}^2}{u_x^2} + \frac{1}{2}\frac{u_t^2}{u_x} - \frac{4}{3}\frac{u_{xx}^2 f(u)}{u_x^3} + \frac{4}{3}\frac{u_{xx}f'(u)}{u_x} \\ &\quad - \frac{5}{3}\frac{u_t f(u)}{u_x^2} + \frac{8}{9}\frac{f(u)^2}{u_x^3} - \frac{4}{9}u_x f''(u). \end{aligned} \tag{30}$$

We next rewrite this symmetry by eliminating u_t, u_{tx}, u_{txx} through the gKN equation (3) and its differential consequences. This provides an explicit classification of all higher-order symmetries (27) up to differential order six (in x derivatives).

Corollary 2.1. *(i) For any $f(u)$, the gKN equation* (3) *admits no symmetries* (27) *of differential order one, two, four, or six.*
(ii) The gKN equation (3) *admits a single symmetry* (27) *of differential order three:*

$$\hat{\mathrm{X}}_1 = \Big(u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x}\Big)\partial_u = u_t\partial_u \tag{31}$$

$$f(u) \text{ arbitrary }. \tag{32}$$

(iii) The gKN equation (3) *admits a single symmetry* (27) *of differential order five:*

$$\begin{aligned} \hat{\mathrm{X}}_2 =& \Big(u_{xxxxx} - 5\frac{u_{xxxx}u_{xx}}{u_x} - \frac{5}{2}\frac{u_{xxx}^2}{u_x} + \frac{25}{2}\frac{u_{xxx}u_{xx}^2}{u_x^2} - \frac{45}{8}\frac{u_{xx}^4}{u_x^3} - \frac{5}{3}\frac{u_{xxx}f(u)}{u_x^2} \\ &+ \frac{25}{6}\frac{u_{xx}^2 f(u)}{u_x^3} - \frac{5}{3}\frac{u_{xx}f'(u)}{u_x} - \frac{5}{18}\frac{f(u)^2}{u_x^3} + \frac{5}{9}u_x f''(u)\Big)\partial_u \end{aligned} \tag{33}$$

$$f'''''(u) = 0 \quad (f(u) = C_1u^4 + C_2u^3 + C_3u^2 + C_4u + C_5). \tag{34}$$

This classification establishes that both the KN and gKN equations do not possess any higher-order symmetries (27) up to differential order six (in x derivatives) that involve t, x explicitly.

The symmetry (33) of differential order five is the first higher-order symmetry in the hierarchy generated by the fourth order recursion operator of the KN equation [20], where the root symmetry for this hierarchy is $\hat{\mathrm{X}} = u_x\partial_u$. All of the symmetries in the hierarchy commute with translations on t, x.

3. Classification of conservation laws

A *conservation law* of the gKN equation (3) is a space-time divergence such that

$$D_tT(t, x, u, u_t, u_x, \ldots) + D_xX(t, x, u, u_t, u_x, \ldots) = 0 \tag{35}$$

holds for all formal solutions $u(t, x)$ of the gKN equation (3). The spatial integral of the conserved density T formally satisfies

$$\frac{d}{dt}\int_{-\infty}^{\infty} T dx = -X\Big|_{-\infty}^{\infty} \tag{36}$$

and so if the spatial flux X vanishes at spatial infinity, then

$$\mathcal{C}[u] = \int_{-\infty}^{\infty} T dx = \text{const.} \tag{37}$$

formally yields a conserved quantity for gKN equation (3). Conversely, any such conserved quantity arises from a conservation law (35). Two conservation laws are equivalent if their conserved densities $T(t, x, u, u_t, u_x, \ldots)$ differ by a total x-derivative $D_x\Theta(t, x, u, u_t, u_x, \ldots)$ on all formal solutions $u(t, x)$, thereby giving the same conserved quantity $\mathcal{C}[u]$ up to boundary terms. Correspondingly, the fluxes $X(t, x, u, u_t, u_x, \ldots)$ of two equivalent conservation laws differ by a total time derivative $-D_t\Theta(t, x, u, u_t, u_x, \ldots)$ on all formal solutions $u(t, x)$. A conservation law is called *trivial* if

$$\begin{aligned} T(t, x, u, u_t, u_x, \ldots) &= \Phi(t, x, u, u_t, u_x, \ldots) + D_x\Theta(t, x, u, u_t, u_x, \ldots) \\ X(t, x, u, u_t, u_x, \ldots) &= \Psi(t, x, u, u_t, u_x, \ldots) - D_t\Theta(t, x, u, u_t, u_x, \ldots) \end{aligned} \tag{38}$$

such that $\Phi = \Psi = 0$ holds on all formal solutions $u(t, x)$. Thus, equivalent conservation laws differ by a trivial conservation law.

The set of all conservation laws (up to equivalence) admitted by the gKN equation (3) forms a vector space on which there is a natural action [13, 21] by the group of all Lie symmetries of the gKN equation (3).

Each conservation law (35) has an equivalent *characteristic form* in which u_t and all derivatives of u_t are eliminated from T and X through use of the gKN equation (3) and its differential consequences. There are two steps to obtaining the characteristic form. First, we eliminate $u_t, u_{tx}, \ldots$ to get

$$\hat{T} = T\big|_{u_t = u_{xxx} - \frac{3}{2}(u_{xx}^2 + f(u))/u_x} = T - \Phi, \quad \hat{X} = X\big|_{u_t = u_{xxx} - \frac{3}{2}(u_{xx}^2 + f(u))/u_x} = X - \Psi \tag{39}$$

so that

$$\big(D_t\hat{T}(t, x, u, u_x, u_{xx}, \ldots) + D_x\hat{X}(t, x, u, u_x, u_{xx}, \ldots)\big)\big|_{u_t = u_{xxx} - \frac{3}{2}(u_{xx}^2 + f(u))/u_x} = 0 \tag{40}$$

where

$$D_t\big|_{u_t=u_{xxx}-\frac{3}{2}(u_{xx}^2+f(u))/u_x} = \partial_t + \Big(u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x}\Big)\partial_u + D_x\Big(u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x}\Big)\partial_{u_x} + \cdots \tag{41}$$

$$D_x\big|_{u_t=u_{xxx}-\frac{3}{2}(u_{xx}^2+f(u))/u_x} = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + \cdots = D_x \tag{42}$$

holds on all formal solutions of the gKN equation (3). Next, moving off of solutions, we use the identity

$$D_t = D_t\big|_{u_t=u_{xxx}-\frac{3}{2}(u_{xx}^2+f(u))/u_x} + \Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big)\partial_u + D_x\Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big)\partial_{u_x} + \cdots . \tag{43}$$

This yields the characteristic form of the conservation law (35)

$$D_t\hat{T}(t,x,u,u_x,u_{xx},\ldots) + D_x\big(\hat{X}(t,x,u,u_x,u_{xx},\ldots) + \hat{\Psi}(t,x,u,u_t,u_x,\ldots)\big) = Q(t,x,u,u_x,u_{xx},\ldots)\Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big) \tag{44}$$

holding identically, where

$$\hat{\Psi} = E_{u_x}(\hat{T})\Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big) + E_{u_{xx}}(\hat{T})D_x\Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big) + \cdots \tag{45}$$

is a trivial flux, and where the function

$$Q = E_u(\hat{T}) \tag{46}$$

is called a *multiplier* (or a *characteristic*). Here $E_u = \partial_u - D_x\partial_{u_x} + D_x^2\partial_{u_{xx}} - \cdots$ denotes the (spatial) Euler operator with respect to u.

From the characteristic equation (44), there is a one-to-one relation between conserved densities (up to equivalence) and multipliers for the gKN equation (3). Note that if a conserved density $\hat{T}$ has differential order $k \geq 0$, then the differential order of the corresponding multiplier Q is at most $2k \geq 0$. For a conserved density $\hat{T}$ of minimal differential order $k \geq 0$, the corresponding multiplier Q has maximal differential order $2k \geq 0$, and from the characteristic equation (44), the flux $\hat{X}$ has differential order $k+2 \geq 2$. We define the *differential order of a conservation law* to be the smallest differential order among all equivalent conserved densities.

Multipliers Q are determined by the condition that their product with the gKN equation is a total space-time divergence. Such divergences have the characterization that their variational derivative with respect to u vanishes identically [13, 19]. This condition

$$\frac{\delta}{\delta u}\bigg(\Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big)Q\bigg) = 0 \tag{47}$$

can be split in an explicit form with respect to $u_t, u_{tx}, u_{txx}, \ldots$, which yields the equivalent equations [22, 23, 24]

$$0 = -D_tQ + D_x^3Q + 3D_x^2\Big(\frac{u_{xx}}{u_x}Q\Big) + D_x\Big(\frac{3}{2}\frac{u_{xx}^2}{u_x^2}Q - \frac{f(u)}{u_x^2}Q\Big) - \frac{f'(u)}{u_x}Q \tag{48}$$

and

$$Q_u = E_u(Q), \quad Q_{u_x} = -E_u^{(1)}(Q), \quad Q_{u_{xx}} = E_u^{(2)}(Q), \quad \ldots \tag{49}$$

holding for all formal solutions $u(t,x)$ of the gKN equation (3). Here $E_u^{(1)} = \partial_{u_x} - 2D_x\partial_{u_{xx}} + 3D_x^2\partial_{u_{xxx}} - \cdots$ and $E_u^{(2)} = \partial_{u_{xx}} - 3D_x\partial_{u_{xxx}} + 6D_x^2\partial_{u_{xxxx}} - \cdots$ denote higher (spatial) Euler operators [13]. These equations (48)–(49) constitute the standard determining system for multipliers (see also Refs.[25, 26]).

The first equation (48) is the adjoint of the symmetry determining equation (7), and its solutions Q are sometimes called *adjoint-symmetries* (or *cosymmetries*). The second equation (49) comprises the Helmholtz conditions, which are necessary and sufficient for Q to be an Euler-Lagrange expression (46). Consequently, multipliers are simply adjoint-symmetries that have a variational form, and the determination of conservation laws via multipliers is a kind of adjoint problem of the determination of symmetries. In this formulation, conserved densities and fluxes can be recovered from multipliers either by [16, 19] directly integrating the relation (46) between Q and $\hat{T}$, or by [13, 23, 24] (see also Refs.[27, 28]) using a homotopy integral formula which expresses $\hat{T}$ in terms of Q.

We will now classify all conservation laws (40) up to differential order three admitted by the gKN equation (3). This classification means finding all conserved densities $\hat{T}$ up to differential order three (in x derivatives) and all fluxes $\hat{X}$ up to differential order five (in x derivatives), or equivalently, all multipliers (46) up to differential order six (in x derivatives). In particular, any such conservation laws that are admitted for special $f(u)$ will be determined.

For computational purposes, it will be simpler to use an alternative formulation of this classification, which arises through expressing the gKN equation in the solved form (11). To proceed, we first state a useful correspondence result.

Lemma 3.1. *For the gKN equation* (3)*, conservation laws up to differential order three in x derivatives*

$$D_tT(t,x,u,u_x,u_{xx},u_{xxx}) + D_xX(t,x,u,u_x,u_{xx},u_{xxx},u_{xxxx},u_{xxxxx}) = 0 \tag{50}$$

are equivalent to conservation laws

$$D_t\widetilde{T}(t,x,u,u_t,u_x,u_{tx},u_{xx}) + D_x\widetilde{X}(t,x,u,u_t,u_x,u_{tx},u_{xx},u_{tt}) = 0 \tag{51}$$

whose differential order in t,x derivatives is at most two (where $\widetilde{T}_{u_{tt}} = 0$).

The proof involves three main steps. First, by eliminating u_{xxx}, u_{xxxx}, u_{xxxxx} through equation (11), we note that the conserved density and the flux in a conservation law (50) are equivalent to a conserved density $\hat{T}(t,x,u,u_t,u_x,u_{xx})$ and a flux $\hat{X}(t,x,u,u_t,u_x,u_{tx},u_{xx},u_{txx})$, both of which have lower differential order. Second, we look at the highest t-derivative terms in the resulting conservation law

$$D_t\hat{T} + D_x\hat{X} = 0. \tag{52}$$

From $D_t\hat{T}$, we get the term $u_{tt}\hat{T}_{u_t}$, which is second order in t derivatives. From $D_x\hat{X}$, we have $u_{txxx}\hat{X}_{u_{txx}} = D_t\Big(u_t - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x}\Big)\hat{X}_{u_{txx}}$ using equation (11), which yields the t-derivative term $u_{tt}\hat{X}_{u_{txx}}$. The conservation law (52) can then hold only if the coefficient of u_{tt} vanishes, $\hat{T}_{u_t} + \hat{X}_{u_{txx}} = 0$. This relation implies

$$\hat{X} = -u_{txx}\hat{T}_{u_t} + \hat{Y} \tag{53}$$

for some expression $\hat{Y}(t, x, u, u_t, u_x, u_{tx}, u_{xx})$. Finally, motivated by the form of the highest derivative term in $\hat{X}$, we subtract a trivial conservation law (38) given by

$$\Theta(t, x, u, u_t, u_x, u_{xx}) = \int \hat{T}_{u_t} du_{xx}. \tag{54}$$

This subtraction produces an equivalent conservation law

$$D_t\widetilde{T} + D_x\widetilde{X} = 0, \quad \widetilde{T} = \hat{T} - D_x\Theta - \Phi, \quad \widetilde{X} = \hat{X} + D_t\Theta - \Psi \tag{55}$$

where

$$\begin{aligned} \Phi &= \Big(u_t - u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x}\Big)\hat{T}_{u_t} \\ D_x\Theta &= u_{xxx}\hat{T}_{u_t} + \int (\hat{T}_{xu_t} + u_x\hat{T}_{uu_t} + u_{xx}\hat{T}_{u_xu_t} + u_{tx}\hat{T}_{u_tu_t})du_{xx} \end{aligned} \tag{56}$$

and

$$\begin{aligned} \Psi &= 0 \\ D_t\Theta &= u_{txx}\hat{T}_{u_t} + \int (\hat{T}_{tu_t} + u_t\hat{T}_{uu_t} + u_{tx}\hat{T}_{u_xu_t} + u_{tt}\hat{T}_{u_tu_t})du_{xx}. \end{aligned} \tag{57}$$

Hence, on solutions $u(t, x)$ of the gKN equation (11), the equivalent conserved density

$$\begin{aligned} \widetilde{T}(t, x, u, u_t, u_x, u_{tx}, u_{xx}) =& \hat{T} - \Big(u_t - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x}\Big)\hat{T}_{u_t} \\ &- \int (\hat{T}_{xu_t} + u_x\hat{T}_{uu_t} + u_{xx}\hat{T}_{u_xu_t} + u_{tx}\hat{T}_{u_tu_t})du_{xx} \end{aligned} \tag{58}$$

still has differential order two in t, x derivatives, while the differential order of the equivalent flux

$$\hat{X}(t, x, u, u_t, u_x, u_{tx}, u_{xx}, u_{tt}) = \hat{Y} + \int (\hat{T}_{tu_t} + u_t\hat{T}_{uu_t} + u_{tx}\hat{T}_{u_xu_t} + u_{tt}\hat{T}_{u_tu_t})du_{xx} \tag{59}$$

is lower by one order in t, x derivatives. This completes the proof of Lemma 3.1.

It is straightforward to derive that a conservation law (51) of mixed differential order two has the characteristic form

$$\begin{aligned} &D_t\widetilde{T}(t, x, u, u_t, u_x, u_{tx}, u_{xx}) + D_x\widetilde{X}(t, x, u, u_t, u_x, u_{tx}, u_{xx}, u_{tt}) \\ &= \widetilde{Q}(t, x, u, u_t, u_x, u_{tx}, u_{xx}, u_{tt})\Big(u_t - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big) \end{aligned} \tag{60}$$

holding as an identity, with the relation

$$\widetilde{Q} = -\widetilde{X}_{u_{xx}}. \tag{61}$$

In this formulation, note that the mixed differential order of the multiplier (61) is two.

The determining system for these conservation law multipliers (61) is given by the variational derivative condition (47), which can be split with respect to u_{xxx}, u_{xxxt}, u_{xxxx}, u_{xxxxt}, u_{xxxtt}, u_{xxxxx}. This splitting yields the adjoint-symmetry equation (48) plus additional Helmholtz-type equations given by

$$\begin{aligned}\hat{E}_u(\widetilde{Q}) = & -\partial_{u_{xx}}\Big(\hat{D}_x\Big(3\frac{u_{xx}}{u_x}\widetilde{Q}\Big) + \Big(\frac{3}{2}\frac{u_{xx}^2}{u_x^2} + \frac{f(u)}{u_x^2}\Big)\widetilde{Q}\Big) + \hat{D}_x^2(\widetilde{Q}_{u_{xx}}) \\ & + \hat{D}_x\Big(\partial_{u_{xx}}\Big(\hat{D}_x\widetilde{Q} - 3\frac{u_{xx}}{u_x}\widetilde{Q}\Big)\Big) + \partial_{u_{xx}}\Big(\hat{D}_x^2\widetilde{Q}\Big)\end{aligned} \tag{62}$$

$$-\hat{E}_u^{(1,t)}(\widetilde{Q}) = 2\hat{D}_x(\widetilde{Q}_{u_{tx}}) + \partial_{u_{tx}}\Big(\hat{D}_x\widetilde{Q} - 3\frac{u_{xx}}{u_x}\widetilde{Q}\Big) \tag{63}$$

$$-\hat{E}_u^{(1,x)}(\widetilde{Q}) = 2\hat{D}_x(\widetilde{Q}_{u_{xx}}) + \partial_{u_{xx}}\Big(\hat{D}_x\widetilde{Q} - 3\frac{u_{xx}}{u_x}\widetilde{Q}\Big) \tag{64}$$

$$\partial_{u_{xx}}\Big(\hat{E}_{u_x}(\widetilde{Q})\Big) = \partial_{u_{xx}}\Big(\hat{D}_x(\widetilde{Q}_{u_{xx}u_{xx}})\Big) + \partial_{u_{xx}}^2\Big(\hat{D}_x\widetilde{Q} - 3\frac{u_{xx}}{u_x}\widetilde{Q}\Big) \tag{65}$$

with $\hat{D}_x$ denoting D_x restricted to solutions $u(t,x)$ of the gKN equation (11). Here $\hat{E}_u = \partial_u - D_t\partial_{u_t} - \hat{D}_x\partial_{u_x} + D_t^2\partial_{u_{tt}} + D_t\hat{D}_x\partial_{u_{tx}} + \hat{D}_x^2\partial_{u_{xx}} - \cdots$ denotes the (full) Euler operator restricted to solutions; $E_u^{(1,t)} = \partial_{u_t} - \hat{D}_x\partial_{u_{tx}} - 2D_t\partial_{u_{tt}} + \hat{D}_x^2\partial_{u_{txx}} + 2D_t\hat{D}_x\partial_{u_{ttx}} + 3D_t^2\partial_{u_{ttt}} - \cdots$ and $E_u^{(1,x)} = \partial_{u_x} - D_t\partial_{u_{tx}} - 2\hat{D}_x\partial_{u_{xx}} + D_t^2\partial_{u_{ttx}} + 2D_t\hat{D}_x\partial_{u_{txx}} + 3\hat{D}_x^2\partial_{u_{xxx}} - \cdots$ denote higher Euler operators restricted to solutions.

Hence, the determining equations for multipliers $\widetilde{Q}(t,x,u,u_t,u_x,u_{tx},u_{xx},u_{tt})$ are comprised by equations (48) and (62)–(65). Computationally, these equations can be solved in a direct way after they are further split into a linear overdetermined system arising from the coefficients of u_{txx}, u_{ttx}, u_{ttt}, u_{tttx}. This leads to the following classification result.

Theorem 3.1. *(i) The gKN equation* (3) *admits no conservation law multipliers* $\widetilde{Q}(t,x,u)$ *of differential order zero and no conservation law multipliers* $\widetilde{Q}(t,x,u,u_t,u_x)$ *of differential order one, for any* $f(u)$.
(ii) For arbitrary $f(u)$*, the gKN equation* (3) *admits a single conservation law multiplier of mixed differential order two:*

$$\widetilde{Q}_1 = \frac{u_{tx}u_x - u_{xx}u_t}{u_x^3}. \tag{66}$$

(iii) Additional conservation law multipliers of mixed differential order two are admitted by the gKN equation (3) *only for the following* $f(u) \not\equiv 0$*:*

(a) $f(u) = C_1u^4 + C_2u^3 + C_3u^2 + C_4u + C_5 \neq \pm(\tilde{C}_1u^2 + \tilde{C}_2u + \tilde{C}_3)^2$

$$\begin{aligned}\widetilde{Q}_{2\mathrm{a}} = & \frac{u_{tx}u_t - u_{tt}u_x}{u_x^3} - \frac{4}{3}\Big(\frac{u_{xx}^3}{u_x^6} + \frac{u_{xx}u_t}{u_x^5} + \frac{u_{tx}}{u_x^4}\Big)f(u) + \frac{8}{9}\frac{u_{xx}}{u_x^6}f(u)^2 \\ & + 2\Big(\frac{u_{xx}^2}{u_x^4} + \frac{2}{3}\frac{u_t}{u_x^3}\Big)f'(u) - \frac{4}{3}\frac{u_{xx}}{u_x^2}f''(u) - \frac{4}{9}\frac{1}{u_x^4}f(u)f'(u) + \frac{4}{9}f'''(u)\end{aligned} \tag{67}$$

$$\widetilde{Q}_{3\mathrm{a}} = 3t\widetilde{Q}_{2\mathrm{a}} - x\widetilde{Q}_1 + \frac{4}{3}\frac{f(u)}{u_x^3} - 2\frac{u_t}{u_x^2} \tag{68}$$

(b) $f(u) = \pm g(u)^2, \quad g(u) = C_1u^2 + C_2u + C_3$

$$\widetilde{Q}_{2\text{b}} = -2\frac{u_{xx}g(u)}{u_x^3} + 2\frac{g'(u)}{u_x} \tag{69}$$

$$\widetilde{Q}_{3\text{b}} = \widetilde{Q}_{2\text{a}}\big|_{f(u)=\pm g(u)^2}, \quad \widetilde{Q}_{4\text{b}} = \widetilde{Q}_{3\text{a}}\big|_{f(u)=\pm g(u)^2} \tag{70}$$

Remark 3.1. *$f(u)$ is the square of a quadratic polynomial iff* $4f(u)^2f'''(u) - 6f(u)f'(u)f''(u) + 3f'(u)^3 = 0$.

Each multiplier $\widetilde{Q}$ determines a conserved density $\widetilde{T}$ and a flux $\widetilde{X}$, up to equivalence. The simplest way to obtain explicit expressions for them is by first splitting the characteristic equation (60) with respect to u_{xxx}, u_{txx}, u_{ttx}, and next integrating the resulting linear system

$$\widetilde{X}_{u_{xx}} + \widetilde{Q} = 0, \quad \widetilde{X}_{u_{tt}} + \widetilde{T}_{u_{tx}} = 0, \quad \widetilde{X}_{u_{tx}} + \widetilde{T}_{u_{xx}} = 0, \tag{71}$$

$$\begin{aligned} &\widetilde{T}_t + u_t\widetilde{T}_u + u_{tt}\widetilde{T}_{u_t} + u_{tx}\widetilde{T}_{u_x} + \widetilde{X}_x + u_x\widetilde{X}_u + u_{tx}\widetilde{X}_{u_t} + u_{xx}\widetilde{X}_{u_x} \\ &= \Big(u_t + \frac{3}{2}\frac{u_{xx}^2}{u_x} - \frac{f(u)}{u_x}\Big)\widetilde{Q}. \end{aligned} \tag{72}$$

This yields the following conserved densities $\widetilde{T}(t, x, u, u_t, u_x, u_{tx}, u_{xx})$ and fluxes $\widetilde{X}(t, x, u, u_t, u_x, u_{tx}, u_{xx}, u_{tt})$:

$$\widetilde{T}_1 = \frac{1}{2}\frac{u_{xx}^2}{u_x^2} + \frac{1}{3}\frac{f(u)}{u_x^2} \tag{73}$$

$$\widetilde{X}_1 = \frac{1}{2}\frac{u_{xx}^2u_t}{u_x^3} - \frac{u_{xx}u_{tx}}{u_x^2} + \frac{1}{2}\frac{u_t^2}{u_x^2} - \frac{1}{3}\frac{u_tf(u)}{u_x^3} \tag{74}$$

$$\widetilde{T}_{2\text{a}} = \frac{1}{2}\frac{u_{xx}^2u_t}{u_x^3} - \frac{u_{xx}u_{tx}}{u_x^2} - \frac{1}{2}\frac{u_t^2}{u_x^2} - \frac{2}{3}\frac{u_{xx}^2f(u)}{u_x^4} + \frac{u_tf(u)}{u_x^3} - \frac{4}{9}\frac{f(u)^2}{u_x^4} + \frac{4}{9}f''(u) \tag{75}$$

$$\begin{aligned} \widetilde{X}_{2\text{a}} =& \frac{1}{2}\frac{u_{tx}^2}{u_x^2} - \frac{u_{xx}u_{tx}u_t}{u_x^3} + \frac{u_{xx}u_{tt}}{u_x^2} + \frac{1}{3}\frac{u_{xx}^4f(u)}{u_x^6} + \frac{2}{3}\frac{u_{xx}^2u_tf(u)}{u_x^5} \\ &+ \frac{4}{3}\frac{u_{xx}u_{tx}f(u)}{u_x^4} - \frac{4}{9}\frac{u_{xx}^2f(u)^2}{u_x^6} - \frac{2}{3}\frac{u_{xx}^3f'(u)}{u_x^4} - \frac{4}{3}\frac{u_{xx}u_tf'(u)}{u_x^3} \\ &+ \frac{2}{3}\frac{u_{xx}^2f''(u)}{u_x^2} + \frac{4}{9}\frac{u_{xx}f(u)f'(u)}{u_x^4} + \frac{1}{3}\frac{u_t^2f(u)}{u_x^4} - \frac{4}{9}\frac{u_tf(u)^2}{u_x^5} \\ &+ \frac{4}{27}\frac{f(u)^3}{u_x^6} + \frac{2}{9}\frac{f'(u)^2 - 2f(u)f''(u)}{u_x^2} - \frac{4}{9}u_{xx}f'''(u) + \frac{2}{9}u_x^2f''''(u) \end{aligned} \tag{76}$$

$$\widetilde{T}_{3\text{a}} = 3t\widetilde{T}_{2\text{a}} - x\widetilde{T}_1 \tag{77}$$

$$\widetilde{X}_{3\text{a}} = 3t\widetilde{X}_{2\text{a}} - x\widetilde{X}_1 - \frac{4}{3}\frac{u_{xx}f(u)}{u_x^3} + 2\frac{u_{xx}u_t}{u_x^2} - \frac{4}{3}\frac{f'(u)}{u_x} \tag{78}$$

$$\widetilde{T}_{2\text{b}} = \frac{g(u)}{u_x} \tag{79}$$

$$\widetilde{X}_{2b} = \frac{u_{xx}^2 g(u)}{u_x^3} - 2\frac{u_{xx}g'(u)}{u_x} + \frac{u_t g(u)}{u_x^2} \mp \frac{2}{3}\frac{g(u)^3}{u_x^3} + 2u_x g''(u) \tag{80}$$

$$\begin{aligned}\widetilde{T}_{3b} =& \frac{1}{2}\frac{u_{xx}^2 u_t}{u_x^3} - \frac{u_{xx}u_{tx}}{u_x^2} - \frac{1}{2}\frac{u_t^2}{u_x^2} \mp \frac{2}{3}\frac{u_{xx}^2 g(u)^2}{u_x^4} \pm \frac{u_t g(u)^2}{u_x^3} - \frac{4}{9}\frac{g(u)^4}{u_x^4} \\ & \pm \frac{8}{9}\big(g(u)g''(u) + g'(u)^2\big)\end{aligned} \tag{81}$$

$$\begin{aligned}\widetilde{X}_{3b} =& \frac{1}{2}\frac{u_{tx}^2}{u_x^2} - \frac{u_{xx}u_{tx}u_t}{u_x^3} + \frac{u_{xx}u_{tt}}{u_x^2} \pm \frac{1}{3}\frac{u_{xx}^4 g(u)^2}{u_x^6} \pm \frac{2}{3}\frac{u_{xx}^2 u_t g(u)^2}{u_x^5} \\ & \pm \frac{4}{3}\frac{u_{xx}u_{tx}g(u)^2}{u_x^4} - \frac{4}{9}\frac{u_{xx}^2 g(u)^4}{u_x^6} \mp \frac{4}{3}\frac{u_{xx}^3 g(u)g'(u)}{u_x^4} \mp \frac{8}{3}\frac{u_{xx}u_t g(u)g'(u)}{u_x^3} \\ & \pm \frac{4}{3}\frac{u_{xx}^2\big(g(u)g''(u) + g'(u)^2\big)}{u_x^2} + \frac{8}{9}\frac{u_{xx}g(u)^3 g'(u)}{u_x^4} \pm \frac{1}{3}\frac{u_t^2 g(u)^2}{u_x^4} - \frac{4}{9}\frac{u_t g(u)^4}{u_x^5} \\ & + \frac{4}{27}\frac{g(u)^6}{u_x^6} - \frac{8}{9}\frac{g(u)^3 g''(u)}{u_x^2} \mp \frac{8}{9}u_{xx}\big(g(u)g'''(u) + 3g'(u)g''(u)\big) \\ & \pm \frac{4}{9}u_x^2\big(g(u)g''''(u) + 4g'(u)g'''(u) + 3g''(u)^2\big)\end{aligned} \tag{82}$$

$$\widetilde{T}_{4b} = 3t\widetilde{T}_{3b} - x\widetilde{T}_1 \tag{83}$$

$$\widetilde{X}_{4b} = 3t\widetilde{X}_{3b} - x\widetilde{X}_1 + 2\frac{u_{xx}u_t}{u_x^2} \mp \frac{4}{3}\Big(\frac{u_{xx}g(u)^2}{u_x^3} + 2\frac{g(u)g'(u)}{u_x}\Big) \tag{84}$$

We now rewrite these conserved densities and fluxes by first eliminating u_t, u_{tx}, u_{tt} through the gKN equation (3) and its differential consequences, and next adding appropriate total derivative terms $D_x\Theta(t, x, u, u_x, u_{xx}, u_{xxx})$ and $-D_t\Theta(t, x, u, u_x, u_{xx}, u_{xxx})$ to obtain equivalent conserved densities and fluxes that have minimal differential order in x derivatives. The results provide an explicit classification of all conservation laws (50) up to differential order three (in x derivatives).

Corollary 3.1. *(i) The gKN equation* (3) *admits no conservation laws* (44) *of differential order zero, for any $f(u)$.*
(ii) The gKN equation (3) *admits a single conservation law* (44) *of differential order one:*

$$T_{2b} = \frac{g(u)}{u_x} \tag{85}$$

$$X_{2b} = \frac{u_{xxx}g(u)}{u_x^2} - \frac{1}{2}\frac{u_{xx}^2 g(u)}{u_x^3} - 2\frac{u_{xx}g'(u)}{u_x} \pm \frac{1}{3}\frac{g(u)^3}{u_x^3} + 2u_x g''(u) \tag{86}$$

with

$$f(u) = \pm g(u)^2, \quad g'''(u) = 0 \quad (g(u) = C_1u^2 + C_2u + C_3). \tag{87}$$

(iii) The gKN equation (3) *admits a single conservation law* (44) *of differential order two:*

$$T_1 = \frac{1}{2}\frac{u_{xx}^2}{u_x^2} + \frac{1}{3}\frac{f(u)}{u_x^2} \tag{88}$$

$$X_1 = -\frac{u_{xxxx}u_{xx}}{u_x^2} + \frac{1}{2}\frac{u_{xxx}^2}{u_x^2} + \frac{2}{3}\frac{u_{xxx}(3u_{xx}^2 + f(u))}{u_x^3} - \frac{9}{8}\frac{u_{xx}^4}{u_x^4} + \frac{1}{2}\frac{u_{xx}^2 f(u)}{u_x^4} - \frac{u_{xx}f'(u)}{u_x^2} + \frac{1}{6}\frac{f(u)^2}{u_x^4} \tag{89}$$

with

$$f(u) \text{ arbitrary }. \tag{90}$$

(iv) The gKN equation (3) *admits two conservation laws* (44) *of differential order three:*

$$T_{2\mathrm{a}} = \frac{1}{2}\frac{u_{xxx}^2}{u_x^2} - \frac{3}{8}\frac{u_{xx}^4}{u_x^4} + \frac{5}{6}\frac{u_{xx}^2 f(u)}{u_x^4} + \frac{1}{18}\frac{f(u)^2}{u_x^4} - \frac{5}{9}f''(u) \tag{91}$$

$$\begin{aligned} X_{2\mathrm{a}} = &-\frac{u_{xxxxx}u_{xxx}}{u_x^2} + \frac{1}{2}\frac{u_{xxxx}^2}{u_x^2} + \frac{u_{xxxx}u_{xxx}u_{xx}}{u_x^3} + \frac{1}{6}\frac{u_{xxxx}u_{xx}(9u_{xx}^2 - 10f(u))}{u_x^4} \\ &+ 3\frac{u_{xxx}^3}{u_x^3} - \frac{2}{3}\frac{u_{xxx}^2(9u_{xx}^2 - 2f(u))}{u_x^4} + \frac{1}{9}\frac{u_{xxx}(15u_{xx}^2 + 2f(u))f(u)}{u_x^5} \\ &+ \frac{5}{3}\frac{u_{xxx}u_{xx}f'(u)}{u_x^3} + \frac{9}{8}\frac{u_{xx}^6}{u_x^6} - \frac{29}{12}\frac{u_{xx}^4 f(u)}{u_x^6} - \frac{1}{6}\frac{u_{xx}^3 f'(u)}{u_x^4} \\ &+ \frac{19}{18}\frac{u_{xx}^2 f(u)^2}{u_x^6} - \frac{5}{6}\frac{u_{xx}^2 f''(u)}{u_x^2} - \frac{5}{9}\frac{u_{xx}f(u)f'(u)}{u_x^4} - \frac{4}{9}u_{xx}f'''(u) \\ &+ \frac{1}{27}\frac{f(u)^3}{u_x^6} + \frac{5}{18}\frac{2f(u)f''(u) - f'(u)^2}{u_x^2} + \frac{2}{9}u_x^2 f''''(u) \end{aligned} \tag{92}$$

and

$$T_{3\mathrm{a}} = 3tT_{2\mathrm{a}} - xT_1 \tag{93}$$

$$X_{3\mathrm{a}} = 3tX_{2\mathrm{a}} - xX_1 - \frac{u_{xxx}u_{xx}}{u_x^2} + \frac{2}{3}\frac{u_{xx}f(u)}{u_x^3} + \frac{5}{3}\frac{f'(u)}{u_x} \tag{94}$$

both with

$$f'''''(u) = 0 \quad (f(u) = C_1u^4 + C_2u^3 + C_3u^2 + C_4u + C_5). \tag{95}$$

The multipliers (46) corresponding to these conservation laws (85)–(94) are found to coincide with the multipliers (66)–(70) of mixed differential order two when u_{xxx}, u_{xxxx}, u_{xxxxx} are eliminated through the gKN equation in the solved form (11). From this correspondence, we remark that the conserved densities and fluxes in Corollary 3.1 can be obtained alternatively from the multipliers classified in Theorem 3.1 by applying a homotopy integral formula. In particular, for conserved densities, we have

$$T = (u - x)\int_0^1 Q(t, x, u_{(\lambda)}, u_{(\lambda)t}, u_{(\lambda)x}, u_{(\lambda)tx}, u_{(\lambda)xx})d\lambda, \quad u_{(\lambda)} = \lambda u + (1-\lambda)x \tag{96}$$

which can be checked to reproduce, up to equivalence, the conserved densities (85), (88), (91), (93). Note that we have chosen $u_{(0)} = x$ so that, for each multiplier, $Q(t, x, u_{(0)}, u_{(0)t}, u_{(0)x}, u_{(0)tx}, u_{(0)xx}) = Q(t, x, x, 0, 1, 0, 0)$ is a non-singular function of t, x.

The conserved density (88) agrees with the first Hamiltonian for the KN equation [20, 29]. Since this conserved density is also admitted by the gKN equation for arbitrary $f(u)$, we conclude that the first Hamiltonian structure of the KN equation carries over to the gKN equation:

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{f(u)}{u_x} = \mathcal{H}\big(\delta H_1/\delta u\big) \tag{97}$$

where

$$H_1 = -T_1 = -\frac{1}{2}\frac{u_{xx}^2}{u_x^2} - \frac{1}{3}\frac{f(u)}{u_x^2} \tag{98}$$

is the Hamiltonian density, and where

$$\mathcal{H} = u_x D_x^{-1} u_x \tag{99}$$

is a Hamiltonian operator. In the case of the KN equation itself, this Hamiltonian density is the first conserved density in a hierarchy generated by the adjoint of the fourth order recursion operator of the KN equation [10]. The hierarchy is produced by acting with this adjoint operator on multipliers (more generally, on adjoint-symmetries), starting from the multiplier $Q = E_u(H_1)$ corresponding to the Hamiltonian density H_1. The conserved density (91) is precisely the second density in this hierarchy. All of these densities are invariant under translations on t, x.

The conserved density (93) that explicitly involves t, x is new and does not seem to be connected with a recursion operator for the KN equation. The other conserved density (85) is also new, but it is admitted only in the quadratic case of the KN equation.

4. Action of point symmetries on conserved densities

Point symmetries have a natural action on conservation laws. In terms of the infinitesimal generator (4) of a point symmetry, its action on the conserved density T and flux X in a conservation law (35) of the gKN equation (3) is given by

$$\begin{aligned} \widetilde{T} &= \mathrm{prX}(T) + (D_x\xi + D_t\tau)T - TD_t\tau - XD_x\tau \\ \widetilde{X} &= \mathrm{prX}(X) + (D_x\xi + D_t\tau)X - TD_t\xi - XD_x\xi. \end{aligned} \tag{100}$$

This action produces a conserved density $\widetilde{T}$ and a flux $\widetilde{X}$, satisfying $D_t\widetilde{T} + D_x\widetilde{X} = 0$ for all formal solutions $u(t, x)$ of the gKN equation (3). Through the relation (46), it is straightforward to show that the corresponding point symmetry action on multipliers is given by

$$\widetilde{Q} = E_u(\widetilde{T}) = E_u((\eta - \tau u_t - \xi u_x)Q), \quad Q = E_u(T). \tag{101}$$

This formula (101) can be used to determine the transformed conserved density $\widetilde{T}$, since there is a one-to-one correspondence between multipliers and conserved densities modulo trivial densities. In particular, the transformed conserved density $\widetilde{T}$ is trivial (38) if (and only if) $\widetilde{Q} = 0$ holds identically.

From Theorem 2.1, the point symmetries admitted by the gKN equation fall into two different classes. The first class holds for arbitrary $f(u)$ and consists of only translations (19) on t, x. The second class comprises six different forms (20)–(25) of $f(u)$, as derived from the ODE (18). Similarly, from Corollary 3.1, the conserved densities admitted by the gKN equation fall into three classes, one holding when $f(u)$ is arbitrary, another holding when $f(u)$ is a general quartic polynomial (95), and the other holding when $f(u)$ is the square of a general quadratic polynomial (87).

To begin, we specialize the point symmetries in the class (20)–(25) to the cases when $f(u)$ is either a general quartic polynomial or the square of a quadratic polynomial.

For the first case, we have:

$$(\mathrm{a}') \qquad \begin{aligned} f(u) &= C(u+a)^k, \quad k=0,1,2,3 \\ X_{3\mathrm{a}'} &= \tfrac{3}{4}(2-k)t\partial_t + \tfrac{1}{4}(2-k)x\partial_x + (u+a)\partial_u \end{aligned} \tag{102}$$

$$(\mathrm{b}') \qquad \begin{aligned} f(u) &= C(u+a)^4 \\ X_{3\mathrm{b}'} &= -(3t/2)\partial_t - (x/2)\partial_x + (u+a)\partial_u, \\ X_{4\mathrm{b}'} &= (u+a)^2\partial_u \end{aligned} \tag{103}$$

$$(\mathrm{c}') \qquad \begin{aligned} f(u) &= C(a+b+u)^k(a-b-u)^{4-k}, \quad a \neq 0, \quad k=1,2 \\ X_{3\mathrm{c}'} &= (3a(k-2)t/2)\partial_t + (a(k-2)x/2)\partial_x + ((u+b)^2-a^2)\partial_u \end{aligned} \tag{104}$$

Note class (c′) is preserved under $k \to 4-k$, $a \leftrightarrow -a$.

For the second case, we write $f(u) = g(u)^2$. Then we have:

$$(\mathrm{a}'') \qquad \begin{aligned} g(u) &= C(u+a) \\ X_{3\mathrm{a}''} &= X_{3\mathrm{a}'}|_{k=2} = (u+a)\partial_u \end{aligned} \tag{105}$$

$$(\mathrm{b}'') \qquad \begin{aligned} g(u) &= C(u+a)^2 \\ X_{3\mathrm{b}''} &= X_{3\mathrm{b}'} = -(3t/2)\partial_t - (x/2)\partial_x + (u+a)\partial_u \\ X_{4\mathrm{b}''} &= X_{4\mathrm{b}'} = (u+a)^2\partial_u \end{aligned} \tag{106}$$

$$(\mathrm{c}'') \qquad \begin{aligned} g(u) &= C(a^2-(b+u)^2), \quad a \neq 0 \\ X_{3\mathrm{c}''} &= X_{3\mathrm{c}'}|_{k=2} = ((u+b)^2-a^2)\partial_u \end{aligned} \tag{107}$$

By applying the formula (101), we now obtain the results summarized in Tables 1–3.

	X_1	X_2	X_{3a}	X_{3b}	X_{3c}	X_{3d}	X_{3e}	X_{3f}	X_{4f}
T_1	0	0	aT_1	$-aT_1$	$-aT_1$	$-aT_1$	aT_1	$\frac{1}{2}T_1$	0
$f(u)$	arb.	arb.	(20)	(21)	(22)	(23)	(24)	(25)	(25)

Table 1. Action of point symmetries on conserved density (88) of the gKN equation (3)

5. Concluding remarks

For the KN equation (1)–(2), we have found a new conservation law (93)–(94) that explicitly involves t, x. The conserved density (93) in this conservation law has a simple expression in terms of the first two Hamiltonian densities (88) and (91) of the KN equation:

$$T = 3tH_2 + xH_1 \tag{108}$$

	X_1	X_2	$X_{3a'}$	$X_{3b'}$	$X_{4b'}$	$X_{3c'}$
T_{2a}	0	0	$\frac{3}{4}(k-2)T_{2a}$	$\frac{3}{2}T_{2a}$	0	$\frac{1}{2}a(2-k)T_{2a}$
T_{3a}	$3T_{2a}$	$-T_1$	0	0	0	0
$f(u)$	arb.	arb.	(102)	(103)	(103)	(104)

Table 2. Action of point symmetries on conserved densities (91) and (93) of the gKN equation (3)

	X_1	X_2	$X_{3a''}$	$X_{3b''}$	$X_{4b''}$	$X_{3c''}$
T_{2b}	0	0	0	$-T_{2b}$	0	0
$f(u)$	arb.	arb.	(105)	(106)	(106)	(107)

Table 3. Action of point symmetries on conserved density (85) of the gKN equation (3)

where

$$H_1 = -\frac{1}{2}\frac{u_{xx}^2}{u_x^2} - \frac{1}{3}\frac{p(u)}{u_x^2}, \tag{109}$$

$$H_2 = \frac{1}{2}\frac{u_{xxx}^2}{u_x^2} - \frac{3}{8}\frac{u_{xx}^4}{u_x^4} + \frac{5}{6}\frac{u_{xx}^2 p(u)}{u_x^4} + \frac{1}{18}\frac{p(u)^2}{u_x^4} - \frac{5}{9}p''(u). \tag{110}$$

This expression (108) is like the conserved density involving t, x for the KdV equation, which is related to Galilean invariance and motion of center of mass for KdV solutions. We can interpret the conserved density (108) in the following similar way for solutions of the KN equation.

Consider the conserved quantities (37) defined from the Hamiltonian densities (109) and (110). The first Hamiltonian density yields the Hamiltonian energy

$$\mathcal{E}_1 = \int_{-\infty}^{\infty} H_1 dx \tag{111}$$

which is a constant of the motion for all solutions $u(t, x)$ of the KN equation having sufficiently rapid spatial decay as $x \to \pm\infty$. Similarly the second Hamiltonian density yields a higher-order energy

$$\mathcal{E}_2 = \int_{-\infty}^{\infty} H_2 dx. \tag{112}$$

The conserved quantity defined from the new density (108) is then given by

$$\mathcal{C} = \int_{-\infty}^{\infty} (3tH_2 + xH_1)dx = 3t\mathcal{E}_2 + \mathcal{X}_1(t) \tag{113}$$

where

$$\mathcal{X}_1(t) = \int_{-\infty}^{\infty} xH_1 dx \tag{114}$$

is the center of energy (or first x-moment of energy) for solutions $u(t, x)$. Since $\mathcal{C}$ is a constant of motion for solutions $u(t, x)$ of the KN equation that have sufficiently rapid spatial decay as $x \to \pm\infty$, we conclude $\mathcal{C} = \mathcal{C}|_{t=0} = \mathcal{X}_1(0)$. This yields the relation

$$\mathcal{X}_1(t) = -3t\mathcal{E}_2 + \mathcal{X}_1(0) \tag{115}$$

which expresses the property that the center of energy of solutions moves at a constant speed (equal to $-3\mathcal{E}_2$).

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