Exact solutions of semilinear radial Schrödinger equations by group foliation reduction

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AMMCS 2015, Session SS-RALSMCL
Waterloo, 10 June 2015

Outline

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Exact solutions are of interest for understanding

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as well as for testing numerical solution methods.

General Thoughts on Finding special Solutions

Given: a non-linear PDE for a function $f = f(x, y, z)$ which has x, y, z and f and f_x, f_y, f_z occuring explicitly in the PDE. Which ansatz may have a chance?

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What other simple cut/ansatz becomes possible in the presence of a (point-)symmetry?

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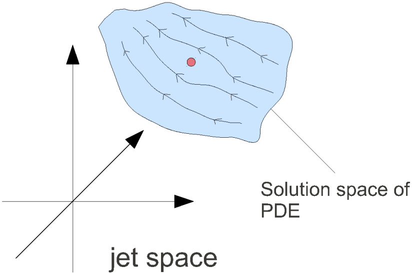
- Group foliation is a geometrical generalization of symmetry reduction.

Symmetry Reduction



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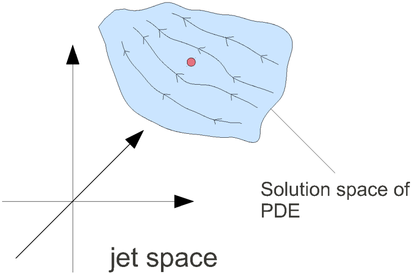

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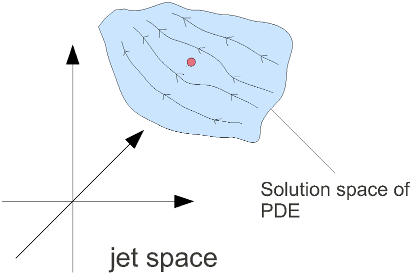

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- equation for $\mathcal{G}$-invariant solutions of PDE
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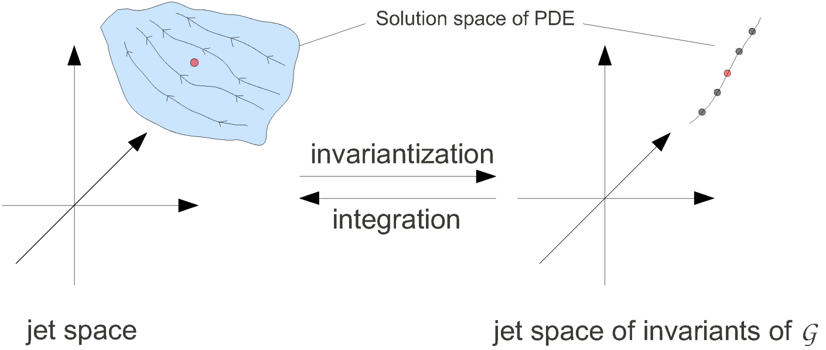
- n^{th} order PDE reduces to n^{th} order ODE iff dim $\mathcal{G}$ is sufficiently large

Group Foliation



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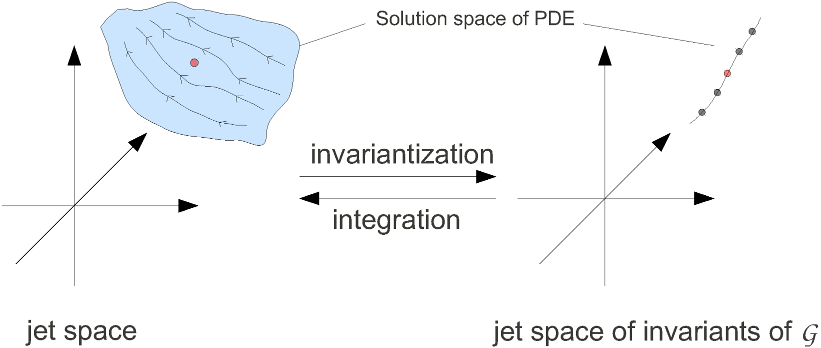

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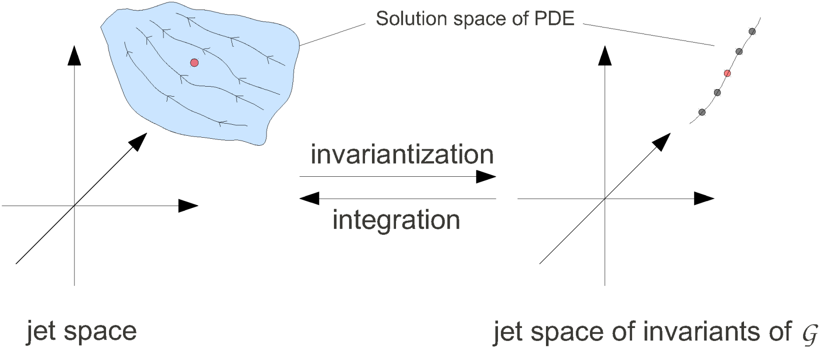

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- equations for $\mathcal{G}$-closed solution families
  - differential order is reduced
  - size of jet space stays same
- n^{th} order PDE converts into $(n-1)^{\text{th}}$ order system of PDEs
- How can one solve the $\mathcal{G}$-invariant system?

Outline

Step 0: Determination of Symmetries

Consider 2$^{\text{nd}}$ order PDE in 2 independent variables and 1 dependent variable

$$F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0$$

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Lie symmetry group G with dim $\mathcal{G} < \infty$
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such that pr $\mathrm{X}_{\mathcal{G}} F = 0$ modulo $F = 0, D_x F = 0, D_t F = 0, \ldots$

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Consider one-dimensional subgroup $\mathcal{G}_1 \in \mathcal{G}$ generated by

$$\mathrm{X} = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$$

Assume prolonged action on jet space $J^\infty = (t, x, u, u_t, u_x, \ldots)$ is regular and transitive.

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Construct group foliation in 5 main steps:

Step 1: Invariantize Coordinates in Jet Space

invariants of X: $y(t, x, u), v(t, x, u)$ s.t. $\mathrm{X}\, y = \mathrm{X} v = 0$
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coordinate transformation in jet space

$$J^{\infty} = (z, y, v, v_z, v_y, v_{yy}, v_{yz}, v_{zz}, ...)$$

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symmetry generator X$= \partial_z \quad \Leftrightarrow \quad \varepsilon$-translation

$v_y, v_z(t,x,u,u_t,u_x)$: 1^{st} order differential invariants of pr X.
$v_{yy}, v_{yz}, v_{zz}(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})$: 2^{nd} order differential invariants of pr X.

etc.

Example: Nonlinear heat equation

$$u_t = u_{xx} + \frac{m}{x} u_x + k u^{p+1} \quad p \neq 0, -1, \quad k \neq 0$$

$m =$ non-negative integer $\Rightarrow$ $m + 1$ dim. radial heat conduction
$m \neq$ non-negative integer $\Rightarrow$ 2 dim. radial heat conduction
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invariants $\zeta(t,x,u)$ s.t. $\mathrm{X}\zeta = 0 = 2t\zeta_t + x\zeta_x - \frac{2}{p} u \zeta_u$

$\Rightarrow \zeta$ is function of $y = \frac{x^2}{t}, \; v = x^{2/p} u$

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canonical coordinate $z(t,x,u)$ s.t. $\quad \mathrm{X}z = 1$
$\Rightarrow z = \ln x + (\text{function of } \; y, v) = \ln x \quad$ (for simplicity)

Example Continued

Change of variables $(t, x, u) \to (z, y, v)$

$$\begin{aligned} x &= e^z \\ t &= \frac{e^{2z}}{y} \\ u &= e^{-\frac{2}{p}z} v \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} D_x &= z_x D_z + y_x D_y &&= e^{-z} D_z + 2e^{-z} y D_y \\ D_t &= z_t D_z + y_t D_y &&= -e^{-2z} y^2 D_y \end{aligned}$$

symmetry generator becomes X$= \partial_z$ translation

Step 2: Invariantize Solution Space of PDE

Each orbit of symmetry group $\mathcal{G}_1$ represents a one-parameter family of solutions $u = u(t, x, c_1)$ satisfying

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action of $\mathcal{G}_1$ on solution is $z \to z + \varepsilon$ in terms of group parameter ε
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PDE is invariant w.r.t. $\mathrm{X} = \partial_z$
$\Leftrightarrow \tilde{F}(y, v, v_y, v_z, v_{yy}, v_{yz}, v_{zz}) = 0 \quad (\tilde{F}_z = XF = 0)$
is the invariantized PDE
solution family satisfies $\tilde{F}(y, v, v_y, v_z, v_{yy}, v_{yz}, v_{zz}) = 0$

Example Continued

The “invariantized” heat equation becomes

$$\begin{aligned} 0 &= v_{zz} + 4yv_{yz} + \left(m - 1 - \frac{4}{p}\right) v_z \\ &\quad + 4y^2 v_{yy} + y\left(y - \frac{8}{p} + 2(m+1)\right) v_y \\ &\quad + \frac{2}{p}\left(1 + \frac{2}{p} - m\right) v + kv^{p+1} \end{aligned} \tag{1}$$

Any solution $v = v(z, y)$ gives a solution $u = x^{-2/p} v(\ln x + c_1, x^2/t)$.

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\end{aligned}
$$

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The method of symmetry reduction (of the number of variables) assumes $v_z = 0$. What remains of (1) has no point symmetries according to LIEPDE and no first integrals according to CONLAW.
$\Rightarrow$ Classical symmetry method reaches a dead end!

Step 3: Adapt Variables to Orbits of Symmetry Group

along orbit $v = v(z + \tilde{c}_1, y)$
$\Rightarrow z = Z(y, v) - \tilde{c}_1$ by implicit function theorem
$\Rightarrow$ use y, v (invariants of X) as independent variables and
use differential invariants of pr X as dependent variables

$$\left.\begin{array}{lcl} v_z|_{\text{orbit}} &=& v_z|_{z=Z-\tilde{c}_1} &=:& \Gamma^{1,0}(y,v) \\ v_y|_{\text{orbit}} &=& v_y|_{z=Z-\tilde{c}_1} &=:& \Gamma^{0,1}(y,v) \end{array}\right\} 1^{\text{st}}\text{order}$$

$$\left.\begin{array}{lcl} v_{zz}|_{\text{orbit}} &=& v_{zz}|_{z=Z-\tilde{c}_1} &=:& \Gamma^{2,0}(y,v) \\ v_{zy}|_{\text{orbit}} &=& v_{zy}|_{z=Z-\tilde{c}_1} &=:& \Gamma^{1,1}(y,v) \\ v_{yy}|_{\text{orbit}} &=& v_{yy}|_{z=Z-\tilde{c}_1} &=:& \Gamma^{0,2}(y,v) \end{array}\right\} 2^{\text{nd}}\text{order}$$

etc.
relations between 1^{st} order Γ's and 2^{nd} order Γ's:

$$(v_z)_z = v_{zz}, \;\; (v_y)_y = v_{yy}, \;\; (v_y)_z = (v_z)_y$$

are called *syzygys*

Computation of Syzygys

$$\left.\begin{aligned} D_z &= \mathrm{pr}\,\partial_z = \partial_z + v_z\partial_v + v_{zz}\partial_{v_z} + v_{zy}\partial_{v_y} + \dots \\ D_y &= \mathrm{pr}\,\partial_y = \partial_y + v_y\partial_v + v_{zy}\partial_{v_z} + v_{yy}\partial_{v_y} + \dots \end{aligned}\right\} \begin{array}{l}\text{prolongations}\\ \text{to } J^\infty\end{array}$$

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evaluate along orbits of $\mathcal{G}_1$

$$\begin{aligned} D_z|_{\text{orbit}} &= 0 + \Gamma^{1,0}\partial_v + \Gamma^{2,0}\partial_{\Gamma^{1,0}} + \Gamma^{1,1}\partial_{\Gamma^{0,1}} + \dots \equiv \hat{D}_z \\ D_y|_{\text{orbit}} &= \partial_y + \Gamma^{0,1}\partial_v + \Gamma^{1,1}\partial_{\Gamma^{1,0}} + \Gamma^{0,2}\partial_{\Gamma^{0,1}} + \dots \equiv \hat{D}_y \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} \Gamma^{2,0} &= \hat{D}_z\Gamma^{1,0} = \Gamma^{1,0}{\Gamma^{1,0}}_v \\ \Gamma^{0,2} &= \hat{D}_y\Gamma^{0,1} = {\Gamma^{0,1}}_y + \Gamma^{0,1}{\Gamma^{1,0}}_v \\ \Gamma^{1,1} &= \hat{D}_z\Gamma^{0,1} = \Gamma^{1,0}{\Gamma^{0,1}}_v \\ &= \hat{D}_y\Gamma^{1,0} = {\Gamma^{1,0}}_y + \Gamma^{0,1}{\Gamma^{1,0}}_v \end{aligned}\right\} \text{ syzygys}$$

etc.

$$J^\infty|_{\text{orbit}} = (y, v, \Gamma^{1,0}, \Gamma^{0,1}, \Gamma^{2,0}, \Gamma^{1,1}, \Gamma^{0,2}, \dots) \quad \text{modulo syzygys}$$

Step 4: Convert Invariantized PDE into 1st Order System

$$\left.\begin{array}{ll}\text{independent variables:} & y, v \\ \text{dependent variables:} & \Gamma^{1,0}, \Gamma^{0,1}\end{array}\right\} \text{ along orbits of } \mathcal{G}_1$$

syzygy relating 1st order Γ's: $0 = {\Gamma^{1,0}}_y + \Gamma^{0,1}{\Gamma^{1,0}}_v - \Gamma^{1,0}{\Gamma^{0,1}}_v$ (2)

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invariantized PDE:

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Substitution of $\Gamma^{2,0}, \Gamma^{1,1}, \Gamma^{0,2}$ using above syzygies gives

$$0 = \hat{F}(y, z, \Gamma^{1,0}, \Gamma^{0,1}, {\Gamma^{1,0}}_y, {\Gamma^{0,1}}_y, {\Gamma^{1,0}}_v, {\Gamma^{0,1}}_v). \tag{3}$$

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(2), (3) are the group-resolving system which is a 1^{st} order system of PDEs for $\Gamma^{1,0}(y, v)$, $\Gamma^{0,1}(y, v)$.

Example Continued

$$v_z|_{\mathrm{orbit}} = \Gamma^{1,0}, \quad \ldots \quad , v_{yy}|_{\mathrm{orbit}} = \Gamma^{0,1}{}_y + \Gamma^{0,1}\Gamma^{1,0}{}_v \Rightarrow$$

$$\begin{aligned} 0 &= (v_{zz} + \ldots + kv^{p+1})|_{\mathrm{orbit}} \text{ (invariantized heat equation)} \\ &= \Gamma^{1,0}{}_v\Gamma^{1,0} + 4y\Gamma^{0,1}{}_v\Gamma^{1,0} + \left(m - 1 - \frac{4}{p}\right)\Gamma^{1,0} \\ &\quad +4y^2(\Gamma^{0.1}{}_y + \Gamma^{0,1}{}_v\Gamma^{0,1} + y\left(y - \frac{8}{p} + 2(m+1)\right)\Gamma^{0,1} \\ &\quad +\frac{2}{p}\left(1 + \frac{2}{p} - m\right)v + kv^{p+1} \end{aligned}$$

Example Continued

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&\quad + 4y^2({\Gamma^{0.1}}_y + {\Gamma^{0,1}}_v \Gamma^{0,1} + y\left(y - \frac{8}{p} + 2(m+1)\right)\Gamma^{0,1} \\
&\quad + \frac{2}{p}\left(1 + \frac{2}{p} - m\right)v + kv^{p+1}
\end{aligned}$$

Using the syzygy

$$\Gamma^{0,1}{\Gamma^{1,0}}_v - \Gamma^{1,0}{\Gamma^{0,1}}_v + \Gamma^{1,0}_y = 0 \tag{4}$$

the scaling group resolving system for $\Gamma^{1,0}(y, v), \Gamma^{0,1}(y, z)$ is ...

Example: Group Resolving Equations

$$\Gamma^{0,1}\Gamma^{1,0}{}_v - \Gamma^{1,0}\Gamma^{0,1}{}_v + \Gamma^{1,0}{}_y = 0 \tag{5}$$

$$\begin{aligned}
& -\frac{1}{2}(2y\Gamma^{0,1} - \Gamma^{1,0})(2y\Gamma^{0,1}{}_v - \Gamma^{1,0}{}_v) - 4y^2\Gamma^{0,1}{}_y + 2y\Gamma^{1,0}{}_y \\
& +\Gamma^{0,1} - (2p+m-1)\Gamma^{1,0} + (2p+m-3)2y\Gamma^{0,1} \qquad (6)\\
= \quad & kv^{p+1} + p(p+m-1)v
\end{aligned}$$

L.h.s. of (5) has general form $\Upsilon_1(\Gamma) := \alpha_1\Gamma \wedge \Gamma_v + \beta_1\Gamma_y$
L.h.s. of (6) has general form $\Upsilon_2(\Gamma) := \alpha_2\Gamma \odot \Gamma_v + \beta_2\Gamma_y + \gamma_2\Gamma$
($\wedge$: antisymmetric product, $\odot$: symmetric product)

Step 5: After solving the System: Reconstruct the PDE Solution Families from Orbits

Let

$$\Gamma^{1,0} = g(y, v), \quad \Gamma^{0,1} = h(y, v)$$

satisfy the group-resolving system.

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on orbit: $v_z = g(y, v), \; v_y = h(y, v)$
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called automorphic property

Outline

1. Case of Integration

- $g = 0$

  $\Rightarrow v_z = 0 \Rightarrow 1^{\text{st}}$ order ODE $v_y = h(y, v)$ for $v(y)$
  (without guarantee that this ODE can be solved)
  any solution $v = v(y, c_1)$ is invariant w.r.t. $X = \partial_z$,

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  $\Rightarrow$ this case is equivalent to the symmetry method

2. Case of Integration

- $g \neq 0$

  on orbit: $v_z = g(y, v),\ v_y = h(y, v)$
  $\Rightarrow$ use hodograph transformation on z, v
  $\Rightarrow$ $z(y, v)$ satisfies

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  solve by line integral formula

  $$z + \tilde{c}_1 = \int \frac{1}{g(y, v)} dv - \frac{h(y, v)}{g(y, v)} dy \quad \text{(path - independent)}$$

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  change of variables $(z,y,v) \to (t,x,u)$
  $\Rightarrow$ solution $u = u(t,x,c_1)$ closed family w.r.t. $\mathcal{G}_1$, i.e. one-dimensional orbit of $\mathcal{G}_1$

Theorem

For 2^{nd} *order PDE*

$$F(t,x,u,u_t,u_x,u_{tt},u_{xx},u_{tx})=0$$

in 2 independent variables t,x *and 1 dependent variable* u *with one-dimensional symmetry (sub-)group* $\mathcal{G}_1$*, solutions of the group-resolving system*

$$\Gamma^{1,0}=g(y,v),\quad \Gamma^{0,1}=h(y,v)$$

are in one-to-one correspondence with one-parameter families of solutions $u=u(t,x,c_1)$ *of the PDE such that the family is closed under the action of* $\mathcal{G}_1$*.*

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This generalizes to PDEs of higher order, arbitrary # of dependent and independent variables and higher dimensional symmetry group (abelian or solvable).

How to find solutions of the group-resolving system?

- All solutions of original PDE arise from solution space of group-resolving system (including those from symmetry reduction which compose special case).
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  $\Rightarrow$ impose reduction ansatz or condition on system, e.g. $\Gamma^{1,0} = 0$ (1. case in reconstruction step)
  $\Rightarrow$ system reduces to 1^{st} order equation for $\Gamma^{0,1}$
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- if original PDE has additional symmetries inherited by the group-resolving system then symmetry reduction possible
  $\Rightarrow$ yields only group-invariant solutions of original PDE

Reduction Methods for Group-Resolving Systems

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- reduction under hidden symmetries
- Bluman's nonclassical method (invariant surface condition)
  Clarkson's direct method and more general functional separation methods
- (successfully used by us:)
  separation ansatz tailored to certain homogeneity features of group-resolving system
  - yields explicit solutions
  - semi-algorithmic $\Rightarrow$ suited to computer algebra (e.g. Crack/Reduce)
  - used for group-resolving systems coming from semilinear PDEs with power nonlinearities

Example: Homogeneity Property

Ansatz $\Gamma = a(y)v + b(y)v^q$ with $q \neq 1$ gives conditions

```
            q         q              q            q              q
0=a10 *v + v *b10  + v *a01*b10*q - v *a01*b10 - v *a10*b01*q + v *a10*b01
     y           y

          2        q               p    2      2*q    2    2      2*q
0=4*a10 *v *y + 4*v *b10 *v*y - 2*v *k*v  - 4*v   *b01 *q*y  + 4*v   *b01*b10*q*y
       y                y

      2*q    2        q                  2      q
   - v   *b10 *q - 4*v *a01*b01*(q+1)*v*y  + 2*v *a01*b10*(q+1)*v*y

        q                      q                q                      q
   + 2*v *a10*b01*q*v*y   + 2*v *a10*b01*v*y - v *a10*b10*(q+1)*v + 4*v *b01*m*v*y

        q                 q              q            q              q
   + 8*v *b01*p*v*y - 12*v *b01*v*y + 2*v *b01*v - 2*v *b10*m*v - 4*v *b10*p*v

        q              2  2  2              2              2              2
   + 2*v *b10*v - 4*a01 *v *y  + 4*a01*a10*v *y + 4*a01*m*v *y + 8*a01*p*v *y

             2            2      2  2            2            2          2
   - 12*a01*v *y + 2*a01*v  - a10 *v  - 2*a10*m*v  - 4*a10*p*v  + 2*a10*v

            2      2  2        2
   + 2*m*p*v  + 2*p *v  - 2*p*v
```

1st condition $\rightarrow a10 =$const + ODE

2nd condition has exponents v^2, v^{q+1}, v^{2q}, v^{p+2}

Example:

$\Rightarrow$ 2 cases: $q = p + 1,\ q = p/2 + 1$ with each 4 conditions for 3 functions $a01, b01, b10$ and 3 constants $p, m, c1$, (k is a parameter), for example:

```
0=2*b10   + a01*b10*p + b01*c1*p
       y

       2    2         2  2                                        2          2
0=4*b01 *p*y  + 8*b01 *y  - 4*b01*b10*p*y - 8*b01*b10*y + b10 *p + 2*b10  + 4*k

       2  2                                                                2
0=4*a01 *y  + 4*a01*c1*y - 4*a01*m*y - 8*a01*p*y + 12*a01*y - 2*a01 + c1

                                                    2
   - 2*c1*m - 4*c1*p + 2*c1 - 2*m*p - 2*p  + 2*p

               2               2
0=4*a01*b01*p*y  + 16*a01*b01*y  + 2*a01*b10*p*y - 8*a01*b10*y + 6*b01*c1*p*y

   + 8*b01*c1*y - 8*b01*m*y - 16*b01*p*y + 24*b01*y - 4*b01 - b10*c1*p

   - 4*b10*c1 + 4*b10*m + 8*b10*p - 4*b10
```

To obtain all solutions one can use computer algebra packages for solving nonlinear overdetermined systems of algebraic/differential equations, e.g. the package CRACK.

Outline

Example: Solutions of the group resolving System I

$$\text{(i)} \quad \Gamma^{0,1} = kv^{p+1}, \quad \Gamma^{1,0} = \frac{2}{p}v + \frac{2k}{y}v^{p+1}$$

$$\text{(ii)} \quad \Gamma^{0,1} = 0, \quad \Gamma^{1,0} = \frac{2}{p}v \pm \sqrt{\frac{-2k}{p+2}}\,v^{1+p/2}, \qquad m = 0$$

$$\text{(iii)} \quad \begin{aligned} \Gamma^{0,1} &= \pm(3-m)\sqrt{\frac{k(1-m)}{m-2}}\,v^{\frac{m-2}{m-1}} \\ \Gamma^{1,0} &= 2(1-m)v \pm 2\sqrt{\frac{k(1-m)}{m-2}}\left(\frac{1}{2} + \frac{3-m}{y}\right)v^{\frac{m-2}{m-1}} \\ p &= \frac{2}{1-m} \end{aligned}$$

Example: Solutions of the group resolving System II

$$
\text{(iv)}\quad
\begin{aligned}
\Gamma^{0,1} &= 0\\
\Gamma^{1,0} &= \pm\sqrt{k(1-m)}\,v^{\frac{1}{m-1}} - \frac{(m-1)^2}{m-2}v\\
p &= \frac{4-2m}{m-1}
\end{aligned}
$$

$$
\text{(v)}\quad
\begin{aligned}
\Gamma^{0,1} &= \frac{3}{3y+1}(v \pm \sqrt{-2k}v^2)\\
\Gamma^{1,0} &= \frac{3}{2y(3y+1)}\left(\left(y^2+\frac{5}{3}y+4\right)v\right.\\
&\qquad \left.\pm\sqrt{-2k}\left(y^2+\frac{1}{3}y+4\right)v^2\right)\\
p &= 2,\ m = \frac{3}{2}
\end{aligned}
$$

Example: Solutions of the group resolving System III

$$
\begin{aligned}
\text{(vi)} \quad \Gamma^{0,1} &= \frac{3}{3y+1} v \pm \frac{3}{2}\sqrt{k} v^{-1} \\
\Gamma^{1,0} &= \frac{3}{y(3y+1)} \left(\left(-y^2 + \frac{1}{3} y + 2 \right) v \right. \\
&\qquad \left. \pm \sqrt{k} \left(y^2 + \frac{10}{3} y + 1 \right) v^{-1} \right) \\
p &= -4, \quad m = \frac{3}{2}
\end{aligned}
$$

Solutions of Nonl. Heat Eqn. $u_t = u_{xx} + \frac{m}{x} u_x + k u^{p+1}$

$$\text{(i)} \quad u = (-kp(t+c_1))^{-1/p}$$

invariant under scaling symmetry and time-translation
X$= 2(t+c_1)\partial_t + x\partial_x - \frac{2}{p} u \partial_u$

$$\text{(ii)} \quad u = x^{-2/p} \left(\pm \frac{p}{2} \sqrt{\frac{-2k}{p+2}} \ln x + c_1 \right)^{-2/p}, \quad m = 0$$

non-invariant w.r.t. X$= a\partial_t + b(2t\partial_t + x\partial_x - \frac{2}{p} u\partial_u)$

Solutions of Nonl. Heat Equation continued

$$\text{(iii)} \quad u = \left(\pm\sqrt{\frac{-k}{(m-1)(m-3)}} \left(\frac{x}{2} - (m-3)\frac{t+c_1}{x} \right) \right)^{m-1}$$

$$q = \frac{3}{1-m}, \quad m \neq 1$$

- invariant w.r.t. X$= 2(t+c_1)\partial_t + x\partial_x - \frac{2}{p}u\partial_u$
  scaling+time-translation
- one-dimensional orbit of scaling group

$$(t \to e^{2\varepsilon}, \;\; x \to e^{\varepsilon}x, \;\; u \to e^{-2\varepsilon/q}u) \Rightarrow (c_1 \to \tilde{c}_1 = e^{-\varepsilon}c_1)$$

(ε =group parameter)

Solutions of Nonl. Heat Equation continued

$$\text{(iv)} \quad u = \left(\pm\sqrt{\frac{1-m}{k}}\left(c_1 x^{3-m} - x\right)\right)^{\frac{m-1}{m-2}},$$

$$p = \frac{4-2m}{m-1}$$

non-invariant w.r.t. X$= a\partial_t + b(2t\partial_t + x\partial_x - \frac{2}{p}u\partial_u)$

Solutions of Nonl. Heat Equation continued

$$\text{(v)} \quad u = \pm \frac{5}{\sqrt{-2k}} \frac{3t + x^2}{x(15t + x^2) + c_1 x^{1/2}}, \quad q = 2, \ m = 3/2$$

- non-invariant w.r.t. $X = a\partial_t + b(2t\partial_t + x\partial_x - u\partial_u)$
- one-dimensional orbit of scaling group

$$(t \to e^{2\varepsilon}, \quad x \to e^{\varepsilon} x, \quad u \to e^{-\varepsilon} u) \Rightarrow (c_1 \to \tilde{c}_1 = e^{-1/2\varepsilon} c_1)$$

Solutions of Nonl. Heat Equation continued

$$\text{(vi)}\quad u = \left(\pm\sqrt{k}(1 + c_1(3t + x^2)) \left(\frac{3t}{x} + x \right) \right)^{1/2}, \;\; q = -4, m = 3/2.$$

- non-invariant w.r.t. $\mathrm{X} = a\partial_t + b(2t\partial_t + x\partial_x + \frac{1}{2}u\partial_u)$
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$$(t \to e^{2\varepsilon}t, \;\; x \to e^{\varepsilon}x, \;\; u \to e^{\varepsilon/2}u) \Rightarrow (c_1 \to \tilde{c}_1 = e^{2\varepsilon}c_1)$$

Outline

The Equation

$$\mathrm{i}u_t = u_{rr} + mu_r/r + k|u|^p u, \quad p \neq 0, \quad k \neq 0 \tag{7}$$

for $u(t, r)$, and p, m constant.

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- $m > 0 \in \mathbb{N}$: model for slow modulation of radial waves in a weakly nonlinear, dispersive, isotropic medium in $m + 1$ dimensions (Sulem, Sulem)
- $m = 0$: same, only r is the full-line coordinate
- otherwise can be interpreted as slow modulation of two-dimensional radial waves in a planar, weakly nonlinear, dispersive medium containing a point-source disturbance at the origin, with modulation term $(m - 1)u_r/r$.

Point Symmetries

$$
\begin{array}{lll}
\text{time translation} & \mathbf{X}_{\text{trans.}} & = \partial_t \\
\text{phase rotation} & \mathbf{X}_{\text{phas.}} & = \mathrm{i} u \partial_u - \mathrm{i} \bar{u} \partial_{\bar{u}} \\
\text{scaling} & \mathbf{X}_{\text{scal.}} & = 2t\partial_t + r\partial_r - (2/p) u \partial_u - (2/p) \bar{u} \partial_{\bar{u}} \\
\text{inversion} & \mathbf{X}_{\text{inver.}} & = t^2 \partial_t + tr\partial_r - (2t/p + \mathrm{i} r^2/4) u \partial_u \\
 & & \quad -(2t/p - \mathrm{i} r^2/4) \bar{u} \partial_{\bar{u}} \quad \text{(only for } p = 4/n\text{)}
\end{array}
$$

where **X** is the infinitesimal generator of a one-dimensional group of point transformations acting on $(t, r, u, \bar{u})$. The inversion is called a pseudo-conformal transformation, and the special power for which it exists is commonly called the critical power.

Symmetry Groups

On solutions $u = f(t, r)$ of the radial NLS equation (7), the one-dimensional symmetry groups arising from the 4 generators are given by

$$
\begin{aligned}
u &= f(t-\epsilon, r), \\
u &= \exp(\mathrm{i}\phi) f(t, r), \\
u &= \lambda^{-2/p} f(\lambda^{-2} t, \lambda^{-1} r), \\
u &= (1+\epsilon t)^{-2/p} \exp\left(-\frac{\mathrm{i}\epsilon r^2}{4+4\epsilon t}\right) f\left(\frac{t}{1+\epsilon t}, \frac{r}{1+\epsilon t}\right), \quad p = \frac{4}{n},
\end{aligned}
$$

with group parameters $-\infty < \epsilon < \infty$, $0 < \lambda < \infty$, $0 \leq \phi < 2\pi$.

Resulting ODEs

Examples:

For $p = 4/n > 0$ ("critical case"), blow-up solutions

$$u(t,r) = (T-t)^{-n/2} U(\xi) \exp(\mathrm{i}(\omega + r^2/4)/(T-t)), \quad \xi = r/(T-t),$$

are invariant under a certain pseudo-conformal subgroup in the full symmetry group, where $U(\xi)$ satisfies the complex ODE

$$U'' + (n-1)\xi^{-1} U' + \omega U + k|U|^{4/n} U = 0.$$

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For $p > 4/n > 0$ ("super critical case") a general class of blow-up solutions is believed to asymptotically approach

$$u(t,r) = (T-t)^{-1/p} U(\xi) \exp(\mathrm{i}\omega \ln((T-t)/T)), \quad \xi = r/\sqrt{T-t},$$

which is invariant under a certain scaling subgroup in the full symmetry group of (7), where $U(\xi)$ satisfies the complex ODE

$$U'' + ((n-1)\xi^{-1} - \tfrac{1}{2}\mathrm{i}\xi)U' - (\omega + \mathrm{i}/p)U + k|U|^p U = 0.$$

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Both ODEs are intractable.

Time-translation-group Resolving System I

Obvious invariants: $x = r, \ v = u$ satisfy $\mathbf{X}_{\text{trans.}}\{x, v, \bar{v}\} = 0$ and $\mathbf{X}_{\text{phas.}}x = 0$, $\mathbf{X}_{\text{phas.}}v = \mathrm{i}v$, $\mathbf{X}_{\text{phas.}}\bar{v} = -\mathrm{i}\bar{v}$.

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Obvious differential invariants: $G = u_t,\ H = u_r$ satisfy $\mathbf{X}^{(1)}_{\text{trans.}}G = \mathbf{X}^{(1)}_{\text{trans.}}H = 0$ and $\mathbf{X}^{(1)}_{\text{phas.}}G = \mathrm{i}G$, $\mathbf{X}^{(1)}_{\text{phas.}}H = \mathrm{i}H$, where $\mathbf{X}^{(1)}_{\text{trans.}},\ \mathbf{X}^{(1)}_{\text{phas.}}$ are first-order prolongations.

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$x, v, \bar{v}$ are mutually independent,
G, H are related by $D_r G = D_t H$ and the radial NLS equation

$$\mathrm{i}G - r^{1-n}D_r(r^{n-1}H) = kv^{1+p/2}\bar{v}^{p/2}.$$

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To summarize, $G = G(x, v, \bar{v})$, $H = H(x, v, \bar{v})$ satisfy

$$G_x + HG_v - GH_v + \bar{H}G_{\bar{v}} - \bar{G}H_{\bar{v}} = 0$$
$$\mathrm{i}G - (n-1)H/x - H_x - HH_v - \bar{H}H_{\bar{v}} = kv^{1+p/2}\bar{v}^{p/2}$$

what we call the *time-translation-group resolving system*.

Time-translation-group Resolving System II

Lemma

Phase-equivariant solutions $G = g(x, |v|)v$, $H = h(x, |v|)v$ of the time-translation-group resolving system are in one-to-one correspondence with two-parameter families of solutions $u = u(t, r, c_1)\exp(\mathrm{i}c_2)$ of the radial NLS equation satisfying the time-translation invariance property

$$u(t+\epsilon, r, c_1) = u(t, r, \tilde{c}_1(\epsilon, c_1))\exp(\mathrm{i}\tilde{c}_2(\epsilon, c_2)) \tag{8}$$

(in terms of group parameter ϵ) with $\tilde{c}_1(0, c_1) = c_1$ and $\tilde{c}_2(0, c_2) = 0$, where c_1, c_2 are the constants of integration of the pair of parametric first-order ODEs

$$u_r = h(r, u, \bar{u}), \quad u_t = g(r, u, \bar{u})$$

which are invariant under $X_{\mathrm{trans.}}$ and $X_{\mathrm{phas.}}$.

Time-translation-group Resolving System III

Lemma

There is a one-to-one correspondence between two-parameter families of static *solutions* $u = f(r, c_1)\exp(\mathrm{i}c_2)$ *of the radial NLS equation* (7) *and solutions of the time-translation-group resolving system that satisfy condition* $G = 0$*.*

A Homogeneity Observation

The group-resolving systems for $G = G(x, v, \bar{v})$, $H = H(x, v, \bar{v})$ have the structure

$$\begin{pmatrix} \Upsilon_1(G, H) \\ G + \Upsilon_2(H) \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathrm{i}kv^{1+p/2}\bar{v}^{p/2} \end{pmatrix}$$

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where Υ_1 and Υ_2 are quadratic nonlinear 1st-order differential operators which obey the homogeneity properties:

$$\Upsilon_1(\alpha v + \beta v^b\bar{v}^a, \gamma v + \lambda v^b\bar{v}^a) = \nu v + \mu v^b\bar{v}^a$$
$$\Upsilon_2(\gamma v + \lambda v^b\bar{v}^a) = \nu v + \mu v^b\bar{v}^a + \epsilon v^{2b-1}\bar{v}^{2a} + \kappa v^{a+b}\bar{v}^{a+b-1}$$

with α, β, ϵ, κ, λ, ν, μ denoting functions only of x.

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with α, β, ϵ, κ, λ, ν, μ denoting functions only of x.
Additionally, these operators have the phase invariance properties:

$$\begin{aligned} \mathbf{X}_{\text{phas.}}\Upsilon_1(v^{a+1}\bar{v}^a, v^{b+1}\bar{v}^b) &= \mathrm{i}\Upsilon_1(v^{a+1}\bar{v}^a, v^{b+1}\bar{v}^b) \\ \mathbf{X}_{\text{phas.}}\Upsilon_2(v^{b+1}\bar{v}^b) &= \mathrm{i}\Upsilon_2(v^{b+1}\bar{v}^b) \end{aligned}$$

Ansatz

Based on these homogeneity and phase invariance properties the group-resolving system should have solutions of form

$$\begin{aligned} H &= (h_1(x) + h_2(x)|v|^{2a})v, \\ G &= -\Upsilon_2\left((h_1(x) + h_2(x)|v|^{2a})v\right) - \mathrm{i}kv|v|^p, \end{aligned}$$

$a \neq 0$, satisfying $\mathbf{X}^{(1)}_{\text{phas.}} H = \mathrm{i}H$ and $\mathbf{X}^{(1)}_{\text{phas.}} G = \mathrm{i}G$.

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In particular, the homogeneity properties show that the v term in H will produce terms in $\Upsilon_1(G, H)$ and $\Upsilon_2(H)$ that contain the same powers $v,\ v|v|^{2a}$ already appearing in H and G.

Splitting

Substitution of the ansatz in the group-resolving system gives one equation with monomial powers

$$v, \quad v|v|^{2a}, \quad v|v|^{4a}, \quad v|v|^{6a}, \quad v|v|^{p}, \quad v|v|^{p+2a}.$$

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Each splitting results in an overdetermined differential system for 2 complex (= 4 real) functions of x and constants a, p, m.

Solution of Overdetermined Systems I

Computer algebra package / system: CRACK / REDUCE

Methods: computation of differential Gröbner basis, integrations, splittings, maintaining list of inequalities, $>$ 80 modules, link to external packages SINGULAR and DIFFELIM
allows different levels of automation

Problems: increasing length of equations and large number of cases and sub^n-cases

Solution of Overdetermined Systems continued

Unorthodox measures:

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- reducing the number of different x-dependent functions including x itself by creating homogeneous equations through
  - introducing new functions, e.g. $h_3(x) := xh_2(x)$ for which some equations become x-free
  - combining equations to eliminate inhomogeneous terms

  with the effect of eliminating x automatically when eliminating the functions so that finally one x-dependent function less needs to be eliminated before splitting wrt. the last x-dependent function becomes possible

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- to work at first only with a subset of equations that are homogeneous in some sense,

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Solution of Overdetermined Systems continued

More unorthodox measures:

- to give the reduction of non-linearity a higher weight than the reduction of differential order
- to try integrating equations and by that reducing the number of terms and lowering the differential order resulting in fewer steps in the decoupling process, reducing the length explosion later on
- after the final splitting large polynomial systems for unknown constants remain to be solved, use the package SINGULAR or resultant computing techniques both applicable from within the package CRACK.

Results for the Time+Phase-Translation-Group Resolving System

Solutions exist only in the cases $a = p/2$, $a = p/4$, and $a = 1/n$. For $p \neq 0$ and $n \neq 1$, these solutions are given by:

$$h_1 = h_2 = 0$$

$$h_1 = \mathrm{Re}\, h_2 = 0, \quad (x^{-1}h_2)' = 0, \quad a = 1/n, \quad n \neq 0$$

$$h_1 = (2-n)x^{-1}, \quad \mathrm{Re}\, h_2 = 0, \quad {h_2}^2 = 2k(2-n)/n,$$
$$a = p/4, \quad p = 2/(2-n), \quad n \neq 2$$

$$h_1 = (2-n)x^{-1}, \quad \mathrm{Re}\, h_2 = 0, \quad {h_2}^2 = -k,$$
$$a = p/4, \quad p = 2(3-n)/(n-2), \quad n \neq 2, 3$$

Results continued

$$h_1 = (2-n)x^{-1}, \quad \operatorname{Im} h_2 = 0, \quad {h_2}^2 = (2-n)k,$$
$$a = p/4, \quad p = 2(3-n)/(n-2), \quad n \neq 2, 3$$

$$h_1 = \operatorname{Im} h_2 = 0, \quad h_2' + (n-1)x^{-1}h_2 + k = 0,$$
$$a = -1/2, \quad p = -1$$

$$\operatorname{Im} h_1 = \operatorname{Im} h_2 = 0, \quad h_1' + {h_1}^2 + (n-1)x^{-1}h_1 = 0,$$
$$h_2' + (h_1 + (n-1)x^{-1})h_2 + k = 0, \quad a = -1/2, \quad p = -1$$

$$\operatorname{Im} h_1 = \operatorname{Im} h_2 = 0, \quad x^2 h_1'' + (2x^2 h_1 + (n-1)x)h_1' - (n-1)h_1 = 0,$$
$$h_2' + (h_1 + (n-1)x^{-1})h_2 + k = 0, \quad a = -1/2, \quad p = -1$$

The Solutions for H and G

For $p \neq 0$ and $n \neq 1$, the earlier ansatz yields the following solutions of the time-translation-group resolving system:

$$
\begin{aligned}
H &= 0, G = -\mathrm{i}kv^{1+p/2}\bar{v}^{p/2} \\
H &= \mathrm{i}C_1 x v^{1+1/n}\bar{v}^{1/n}, \\
& G = \mathrm{i}{C_1}^2 x^2 v^{1+2/n}\bar{v}^{2/n} + C_1 n v^{1+1/n}\bar{v}^{1/n} - \mathrm{i}kv^{1+p/2}\bar{v}^{p/2}, \\
& n \neq 0, \quad C_1 \neq 0 \\
H &= (2-n)x^{-1}v \pm \mathrm{i}\sqrt{2k(1-2/n)}v^{(5-2n)/(4-2n)}\bar{v}^{1/(4-2n)}, \\
& G = \pm(4-n)\sqrt{2k(1-2/n)}x^{-1}v^{(5-2n)/(4-2n)}\bar{v}^{1/(4-2n)} \\
& +\mathrm{i}k(1-4/n)v^{(3-n)/(2-n)}\bar{v}^{1/(2-n)}, \\
& p = 2/(2-n), \quad k(1-2/n) > 0, \quad n \neq 2 \\
H &= (2-n)x^{-1}v \pm \mathrm{i}\sqrt{k}v^{(n-1)/(2n-4)}\bar{v}^{(3-n)/(2n-4)}, \\
& G = 0, \quad p = 2(3-n)/(n-2), \quad k > 0, \quad n \neq 2, 3 \\
H &= (2-n)x^{-1}v \mp \sqrt{(2-n)k}v^{(1-n)/(4-2n)}\bar{v}^{(n-3)/(4-2n)}, \\
& G = 0, \quad p = 2(3-n)/(n-2), \quad k(2-n) > 0, \quad n \neq 2, 3
\end{aligned}
$$

More Solutions for H and G

$$
\begin{aligned}
H &= \left(-(k/n)x + C_1 x^{1-n}\right) v^{1/2}\bar{v}^{-1/2}, \quad G = 0, p = -1,\ n \neq 0 \\
H &= x(C_1 - k\ln x)v^{1/2}\bar{v}^{-1/2}, \quad G = 0, \quad p = -1, \quad n = 0 \\
H &= (2-n)(x + C_1 x^{n-1})^{-1}(v + (C_2 + (k/(2n))x^2)v^{1/2}\bar{v}^{-1/2}) \\
&\quad -(k/n)xv^{1/2}\bar{v}^{-1/2}, \quad G = 0, \quad p = -1, \quad n \neq 0, 2 \\
H &= x(x^2 + C_1)^{-1}(2v - (kC_1\ln x + C_2))v^{1/2}\bar{v}^{-1/2}) \\
&\quad -(k/2)xv^{1/2}\bar{v}^{-1/2}, \quad G = 0, \quad p = -1, \quad n = 0 \\
H &= (\ln x + C_1)^{-1}x^{-1}(v + (C_2 + (k/4)x^2)v^{1/2}\bar{v}^{-1/2}) \\
&\quad -(k/2)xv^{1/2}\bar{v}^{-1/2}, \quad G = 0 \\
&\quad p = -1, \quad n = 2
\end{aligned}
$$

More Solutions for H and G

$$
\begin{aligned}
H &= \pm\sqrt{C_1}\left(C_2 J_{|1-n/2|}(\sqrt{C_1}x) + C_3 Y_{|1-n/2|}(\sqrt{C_1}x)\right)^{-1} \times \\
&\quad \Big((C_2 J_{\mp n/2}(\sqrt{C_1}x) + C_3 Y_{\mp n/2}(\sqrt{C_1}x)) \times \\
&\quad (v + (k/C_1) v^{1/2}\bar{v}^{-1/2}) + C_4 x^{-n/2} v^{1/2}\bar{v}^{-1/2}\Big) \\
&\quad G = \mathrm{i} C_1 v, \quad p = -1, \quad \pm(1-n/2) \geq 0, \quad C_1 > 0 \\
H &= \sqrt{C_1}\left(C_2 I_{|1-n/2|}(\sqrt{C_1}x) + C_3 e^{\mathrm{i}\pi|1-n/2|} K_{|1-n/2|}(\sqrt{C_1}x)\right)^{-1} \times \\
&\quad \Big((C_2 I_{\mp n/2}(\sqrt{C_1}x) + C_3 e^{\mp \mathrm{i}\pi n/2} K_{\mp n/2}(\sqrt{C_1}x)) \times \\
&\quad (v - (k/C_1) v^{1/2}\bar{v}^{-1/2}) + C_4 x^{-n/2} v^{1/2}\bar{v}^{-1/2}\Big) \\
&\quad G = -\mathrm{i} C_1 v, \quad p = -1, \quad \pm(1-n/2) \geq 0, \quad C_1 > 0
\end{aligned}
$$

Solutions of the Radial NLS

The radial NLS equation has the following exact solutions arising from the explicit solutions of the time+phase-translation group resolving systems for $n \neq 1$:

$$
\begin{aligned}
u &= (c_2/k)^{1/p} \exp(\mathrm{i}c_1 - \mathrm{i}c_2 t) \\
u &= (c_2 + c_3 t)^{-n/2} \exp\left(\mathrm{i}c_1 - \frac{\mathrm{i}c_3 r^2}{4(c_2 + c_3 t)}\right. \\
&\quad \left. + \frac{2\mathrm{i}k}{c_3(np-2)}(c_2 + c_3 t)^{1-np/2}\right), \\
&\quad p \neq 2/n, \quad n \neq 0, \quad c_3 \neq 0 \\
u &= (c_2 + c_3 t)^{-n/2} \exp\left(\mathrm{i}c_1 - \frac{\mathrm{i}c_3 r^2}{4(c_2 + c_3 t)} - \frac{\mathrm{i}k}{c_3} \ln|c_2 + c_3 t|\right), \\
&\quad p = 2/n, \quad n \neq 0, \quad c_3 \neq 0
\end{aligned}
$$

More Solutions of the Radial NLS

$$
\begin{aligned}
u &= \left(\pm\sqrt{n(n-2)/(2k)}\right)^{2-n}\left((c_2+(n-4)t)/r\right)^{n-2} \\
&\quad \exp\left(\mathrm{i}c_1+\mathrm{i}(1-n/2)r^2/(c_2+(n-4)t)\right), \\
&\quad p=2/(2-n), \quad n(n-2)/k>0, \quad n\neq 2 \\
u &= \left(k(n-3)^2/(2-n)^3\right)^{(2-n)/(6-2n)}\left(r+c_2r^{3-n}\right)^{(2-n)/(3-n)}\times \\
&\quad \exp(\mathrm{i}c_1), \quad p=2(3-n)/(n-2), \quad k(2-n)>0, \quad n\neq 2,3 \\
u &= \left(c_2^2(n-2)^2/k\right)^{(n-2)/(6-2n)}r^{2-n}\times \\
&\quad \exp(\mathrm{i}c_1+\mathrm{i}c_2r^{n-2}), \\
&\quad p=2(3-n)/(n-2), \quad k>0, \quad n\neq 2,3, \quad c_2\neq 0
\end{aligned}
$$

More Solutions of the Radial NLS

$$
\begin{aligned}
u &= \Big(-k/c_6 + r^{1-n/2}\left(c_2 J_{|1-n/2|}(\sqrt{c_6}\,r) + c_3\,Y_{|1-n/2|}(\sqrt{c_6}\,r)\right) \times \\
&\quad \left(1 + c_5 \int_{c_4}^{r} z^{-1}\left(c_2 J_{|1-n/2|}(\sqrt{c_6}\,z) + c_3\,Y_{|1-n/2|}(\sqrt{c_6}\,z)\right)^{-2} dz\right) \\
&\quad \Big) \exp\left(\mathrm{i}c_1 + \mathrm{i}c_6 t\right), \quad p = -1, \quad c_6 > 0 \\
u &= \Big(k/c_6 + r^{1-n/2}\left(c_2 I_{|1-n/2|}(\sqrt{c_6}\,r) + c_3 K_{|1-n/2|}(\sqrt{c_6}\,r)\right) \times \\
&\quad \left(1 + c_5 \int_{c_4}^{r} z^{-1}\left(c_2 I_{|1-n/2|}(\sqrt{c_6}\,z) + c_3 K_{|1-n/2|}(\sqrt{c_6}\,z)\right)^{-2} dz\right) \\
&\quad \Big) \exp\left(\mathrm{i}c_1 - \mathrm{i}c_6 t\right), \quad p = -1, \quad c_6 > 0 \\
u &= \left(-kr^2/(2n) + c_3 r^{2-n} + c_2\right)\exp(\mathrm{i}c_1), \quad p = -1, \quad n \neq 0, 2 \\
u &= \left(-kr^2/4 + c_3 \ln r + c_2\right)\exp(\mathrm{i}c_1), \quad p = -1, \quad n = 2
\end{aligned}
$$

More Solutions of the Radial NLS

$$
\begin{aligned}
u &= (c_2/r)\exp\left(\mathrm{i}c_1 - \mathrm{i}ktr/c_2 + \mathrm{i}k^2t^3/(3c_2^2)\right), \quad p=-1, \quad n=3 \\
u &= \left(c_2/(rt^{1/2})\right)\exp\left(\mathrm{i}c_1 - \mathrm{i}r^2/(4t) - 2\mathrm{i}krt^{3/2}/(5c_2) + \right. \\
&\quad \left. \mathrm{i}k^2t^4/(25c_2^2)\right), \quad p=-1, \quad n=3 \\
u &= (-(k/2)r^2\ln r + c_3r^2 + c_2)\exp(\mathrm{i}c_1), \quad p=-1, \quad n=0 \\
u &= \left((k/8)r^2 + c_3r^6/t^4 + c_2t^2\right)\exp(\mathrm{i}c_1 - \mathrm{i}r^2/(4t)), \\
&\quad p=-1, v \quad n=-4 \\
u &= \Bigg(-(k/c_6)t^2 + (r^3/t)\left(c_2J_3(\sqrt{c_6}r/t) + c_3Y_3(\sqrt{c_6}r/t)\right)\times \\
&\quad \left(1 + c_5\int_{c_4}^{r/t} z^{-1}(c_2J_3(\sqrt{c_6}z) + c_3Y_3(\sqrt{c_6}z))^{-2}\,dz\right)\Bigg) \\
&\quad \exp\left(\mathrm{i}c_1 - \mathrm{i}c_6/t - \mathrm{i}r^2/(4t)\right), \\
&\quad p=-1, \quad n=-4, \quad c_6>0
\end{aligned}
$$

More Solutions of the Radial NLS

$$
\begin{aligned}
u &= \Bigg((k/c_6)t^2 + (r^3/t)(c_2 I_3(\sqrt{c_6}r/t) + c_3 K_3(\sqrt{c_6}r/t)) \times \\
&\quad \left(1 + c_5 \int_{c_4}^{r/t} z^{-1}(c_2 I_3(\sqrt{c_6}z) + c_3 K_3(\sqrt{c_6}z))^{-2}\, dz\right)\Bigg) \times \\
&\quad \exp\left(\mathrm{i}c_1 + \mathrm{i}c_6/t - \mathrm{i}r^2/(4t)\right), \\
&\quad p = -1, \quad n = -4, \quad c_6 > 0 \\
u &= \left(\pm\sqrt{-k(1+3/n)/2}\right)^{-n/2} \left(r + c_2 t^{-1+4/n} r^{2(1-2/n)}\right)^{-n/2} \times \\
&\quad \exp(\mathrm{i}c_1 - \mathrm{i}r^2/(4t)), \quad p = 8/(1 \pm \sqrt{17}) = (\pm\sqrt{17} - 1)/2, \\
&\qquad n = (1 \pm \sqrt{17})/2, \quad kn < 0
\end{aligned}
$$

More Solutions of the Radial NLS

$$
\begin{aligned}
u &= \left(c_2^2(8-3n)/k\right)^{n/4} r^{2-n} t^{-2+n/2} \times \\
&\quad \exp\left(\mathrm{i}c_1 - \mathrm{i}r^2/(4t) + \mathrm{i}c_2 r^{n-2} t^{2-n}\right) \\
&\quad p = 8/(1 \pm \sqrt{17}) = (\pm\sqrt{17}-1)/2, \\
&\quad n = (1 \pm \sqrt{17})/2, \quad k > 0 \\
u &= (-16k)^{-1/3} r^{2/3} (t(1+c_2 t))^{-2/3} \times \\
&\quad \exp(\mathrm{i}c_1 - \mathrm{i}r^2(1+2c_2 t)/(8t(1+c_2 t))), \\
&\quad p = 3, \quad n = 4/3, \quad k < 0
\end{aligned}
$$

Outline

Summary

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Summary

- Results: explicit blow-up solutions of group-invariant form and non-invariant form, dispersive solutions, standing wave solutions, explicit monopole solutions,
- goup foliation + reduction ansatz + intelligent computing power $\Rightarrow$ effective method for finding exact solutions of nonlinear PDEs
- applied successfully to several types of semilinear PDEs:

  Schrödinger eqns. $iu_t = u_{xx} + \frac{m}{x}u_x + k|u|^p u$
  S. Anco, W. Feng, T. Wolf, (J. Math. Anal. Appl. 2015)

  heat eqns. and reaction-diffusion eqns.
  $u_t = u_{xx} + \frac{m}{x}u_x + (q - ku^p)u$
  S. Anco, S. Ali, T. Wolf, (J. Math. Anal. Appl. 2011, SIGMA 2011)

  wave eqns. $u_{tt} = u_{xx} + \frac{m}{x}u_x + ku^{p+1}$
  S. Anco, S. Liu (J. Math. Anal. Appl. 2005)

Future Work

Application to other types of PDEs, e.g. ≥ 3 independent variables, quasilinear, derivative nonlinearities, larger number of symmetries

The End

Thank you!