Structural equations for Killing tensors of arbitrary rank*

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Abstract

An algorithm is given for bringing the equations of monomial first integrals of arbitrary degree of the geodesic motion in a Riemannian space V_n into the form $(F_A)_{;k} = \sum_B \Gamma_{kAB} F_B$. The F_A are the components of a Killing tensor $K_{i_1 \ldots i_r}$ of arbitrary rank r and its symmetrized covariant derivatives. Explicit formulas are given for rank 1,2 and 3. Killing tensor equations in structural form allow the formulation of algebraic integrability conditions and are supposed to be well suited for integration as it is demonstrated in the case of flat space. A method based on integrability conditions being algebraic is given to compute in a single generic point of spacetime numerically the number of non-trivial Killing tensors using numerical values for the Riemann tensor and its derivatives.

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1 Introduction

As symmetries of a Riemannian space are associated with conservation laws of geodesic motion in this space, the investigation of Killing vectors (KV) and Killing tensors (KT) is a central task in General Relativity in classifying and understanding a spacetime.

A test particle moving in a Riemannian space V_n with velocity $u^i,\ i = 1, \ldots, n$ has as a constant of motion $K_{i_1 \ldots i_r} u^{i_1} \ldots u^{i_r}$ if $K_{i_1 \ldots i_r}$ satisfies the conditions ([4])

$$K_{(i_1 \ldots i_r ; i_{(r+1)})} = 0 \tag{1}$$

where () indicates complete symmetrization and [ ] in the following complete antisymmetrization.

But KT's are not only linked to geodesic motion. When motion is not described by geodesics but when the corresponding Hamiltonian H has a potential function V

$$H = \frac{1}{2} g^{ij}(x^k) p_i p_j + V(x^k)$$

and when a first integral F is polynomial in the momentum variables p_i:

$$F = K_s (p)^s + K_{s-1} (p)^{(s-1)} + \ldots + K_0$$

(indices suppressed) with symmetric contravariant tensors K_i of rank $0, 1, .., s$ then K_s and K_{s-1} have to be KT's (see e.g. [20]) which are to be computed first as they enter the equations for the other K_i.

Another source of interest for KT's comes from the fact that the separability of the Hamilton-Jacobi equation

$$g^{ij} \partial_{x^i} W \partial_{x^j} W = E, \ \ g^{ij} = g^{ji}, \ \ 1 \leq i, j \leq n$$

is dependent on the existence of KT's for the metric g_{ij} which was important for the integration of the geodesic equation for the Kerr metric by Carter and which is important in classical mechanics as well ([1], [2], [10], [27]).

Despite of their importance only few proper KT of rank two, very few of rank three and as far as the author is aware of, none of rank higher than three are known. The main problem in solving equations (1) lies in the high computational effort, as we will see below, especially when the rank is higher than two.

With the availability of computer algebra programs [23], [24],[25] for investigating and solving overdetermined PDE systems like (1) it is possible to try solving these equations directly and automatically. This is the approach pursuit by the package `CLASSYM` written by Guy Grebot [26] in the computer algebra system `CLASSI`. `CLASSYM` formulates the equations for KVs, conformal KVs and KTs of rank 2 including their integrability conditions. Then it uses the package `CRACK` to solve them. In this approach it is up to the intelligence of the program `CRACK` to apply the right integrability conditions at the right time to make the whole task feasible. Although methods used for computing a so-called differential Gröbner basis, for integrating total differentials and others, are algorithmic, there is much freedom left in these algorithms, e.g. the choice of the total ordering in computing the differential Gröbner basis. To use this freedom most efficiently is difficult - for humans as well as for programs.

In this paper a different approach is pursuit, one that is completely algorithmic, and not depending on a particular choice of total orderings of derivatives. It involves rewriting the KT equations of arbitrary rank in the form of structural equations

$$(F_A)_{;k} = \sum_B \Gamma_{kAB} F_B \tag{2}$$

which are homogeneous linear first order PDEs expressing all first order derivatives of a set of functions F_A through expressions algebraic in these functions. Because *all* first order derivatives of *all* F_A are given in terms of F_B, the general solution can only contain free constants.

2 Advantage of tensorial structural equations

The advantage of structural equations is, that all integrability conditions of (2) are easily obtained by calculating $(F_A)_{;[kl]}$ at first from (2) with first order derivatives substituted through (2) and also using the RICCI identity (4) (below). The resulting algebraic linear integrability conditions can be differentiated again and first derivatives be substituted to obtain additional linear integrability conditions. This process can be continued until no new functionally independent conditions result [21].

An advantage of having structural equations in tensorial form, i.e. using the covariant derivative in (2), is that they can be easily contracted with the metric tensor or geometrically meaningful vector fields and scalar equations be computed, as there is a good chance that they may become relatively simple, i.e. short or integrable.

Although the computer algebra program `CRACK` ([23], [26]) for solving overdetermined PDE-systems such as (1), does apply integrability conditions, it does not know about the tensor structure of a PDE system. Therefore some advantage is expected to generate integrability conditions outside of `CRACK` in such a way that their tensorial character is preserved.

Why should it be advantageous to investigate structural equations and not to formulate integrability conditions, for example, by generating a differential Gröbner basis?

The equations of a differential Gröbner basis are generated on the basis of an ordering of variables and of functions. This ordering has to be chosen beforehand. The final form of the differential Gröbner basis is dependent on the ordering applied. As many orderings are possible, it is unlikely that, for example, the vanishing of one derivative of one component of the Killing tensor will show up as an equation in a differential Gröbner basis. But knowing about such equations is important in order to integrate the PDE-system.

On the other hand having a formulation of the PDE-system as structural equations available, such an equation would show up. The reason for this is that all first order derivatives of all Killing tensor components and of all their derivatives of order up to r are expressed through expressions in them. The vanishing of some of the derivatives will lead the way of integrations to be done to integrate the system step by step. The price of this order-independence is a large number of mostly long integrability conditions. The most simple case - flat space - as solved below indicates that once all integrability conditions are satisfied it is better to integrate the structural equations first and to impose the symmetries (13), (14) as linear algebraic conditions on the constants of integration afterwards. For Killing vectors the structural equations take the well known form

$$\begin{aligned} K_{i;j} &= \omega_{ij} \\ \omega_{ij;k} &= K_m R^m_{\ kji} \end{aligned}$$

with the Killing vector K_i and the antisymmetric Killing bivector $\omega_{ij} = -\omega_{ji}$ being the F_A of (2).

For the case of rank two Killing tensors Collinson [3] derived integrability conditions equivalent to the ones in this paper although the Killing tensor equations had not been rewritten as a first order system for some F_A to allow writing the integrability conditions as algebraic conditions for these F_A. Hauser and Malhiot [6],[7] introduced new tensors L, M to formulate structural equations for rank two Killing tensors. Their definition differs from ours and is especially well adapted to do investigations in a null tetrad [8]. Using definitions as those of Hauser and Malhiot, structural equations for conformal KT's of rank 2 were given by Weir in [22]. The maximal number of KT's which is attained if and only if V_n is of constant curvature has been found by G. Thompson (see [19],[20]) to be $\frac{(n+r-1)!(n+r)!}{(n-1)!n!r!(r+1)!}$. If $\mathbf{K}$ is a KT of rank r then all derivatives of $\mathbf{K}$ of order greater than r are expressible in terms of derivatives of $\mathbf{K}$ of order less than or equal to r through the solution of a linear algebraic system of equations [16],[20].

The aim of this paper is to give an algorithm for such a calculation in enough detail to be programmable and to give the concrete form of structural equations for lower ranks.

Applying our method the calculation of all rank r KT's of flat space becomes very simple. It has been shown that KT's of spaces with vanishing or constant curvature are reducible ([20]), i.e. they are linear combinations of symmetrized products of Killing vectors. The alternative constructive proof we give in appendix B will provide as a by-product the relation between our structural equations in terms of symmetrized covariant derivatives of the Killing tensor on one hand and the formalism of Hauser and Malhiot using tensors antisymmetric in pairs of indices on the other hand.

3 Notation

To keep formulae better readable we will use a bold letter $\mathbf{K}$ for the Killing tensor whenever possible which is understood throughout the paper to be of rank r, with r indices taking values

$1, \ldots, n$ in an n-dimensional Riemannian space. Covariant derivatives are denoted by $\mathbf{D}$ and a covariant derivative $D_{i_1} D_{i_2} \ldots D_{i_s}$ of order s is also written as $\mathbf{D}^s$.

If a tensor is to be completely symmetrized over a number of indices then the symmetry operator S will be used, i.e. $S_{i_1 \ldots i_r}$ will denote complete symmetrization over the indices $i_1, \ldots, i_r$. Because the structural equations will be formulated in terms of completely symmetrized covariant derivatives of the Killing tensor $\mathbf{K}$, we define tensors $\mathbf{K}^{(r)(s)}$ through

$$\begin{aligned} K^{(r)(s)}_{i_1 \ldots i_{(r+s)}} &:= K_{i_1 \ldots i_r ; (i_{(r+1)} \ldots i_{(r+s)})} \\ &= S_{i_{(r+1)} \ldots i_{(r+s)}} K_{i_1 \ldots i_r ; i_{(r+1)} \ldots i_{(r+s)}}, \quad 0 \leq s \end{aligned} \tag{3}$$

with $\mathbf{K}^{(r)(0)} := \mathbf{K}$.

The sign of the Riemann tensor is fixed through the RICCI identity

$$2W_{i_1 \ldots i_p ; [cd]} = W_{b i_2 \ldots i_p} R^b{}_{i_1 cd} + \ldots + W_{i_1 \ldots i_{(p-1)} b} R^b{}_{i_p cd}. \tag{4}$$

Only for the purpose of proving the equivalence of Killing tensor equations to a system of structural equations we will drop terms from equations which come in only through the RICCI identity. If such terms are dropped we will use '$\simeq$' instead of '$=$'. That means that with equalities of the form $A \simeq B$, where A, B are n'th order differential expressions in components of $\mathbf{K}$ we will express that

$$\begin{aligned} A = B \quad + \quad & \text{terms of lower than } n\text{'th order in } \mathbf{K} \text{ which} \\ & \text{include the Riemann tensor and vanish in flat space.} \end{aligned}$$

4 Structural equations

The purpose of this section is to formulate structural equations that are equivalent to the Killing tensor equations (1) in terms of tensors $\mathbf{K}^{(r)(s)}$.

This proceeds in two steps.

1. At first differential identities are derived expressing the first order derivative $\mathbf{D}\ \mathbf{K}^{(r)(s)}$ in terms of $\mathbf{K}^{(r)(0)}, \ldots, \mathbf{K}^{(r)(s+1)}$ and the Riemann tensor.

2. In the second step the Killing equations and differential consequences of them are expressed as algebraic conditions for the $\mathbf{K}^{(r)(s)}$. By proving $\mathbf{K}^{(r)(r+1)} \simeq 0$ the resulting structural system is shown to have finitely many equations.

To do the first step the covariant derivatives of order $s+1$ of the Killing tensor $\mathbf{D}^{(s+1)}\ \mathbf{K}$ are expressed in terms of symmetrized derivatives $\mathbf{K}^{(r)(0)} \ldots \mathbf{K}^{(r)(s+1)}$ and the Riemann tensor using the RICCI identity (4) and its derivatives

$$2K_{i_1 \ldots ; \ldots i_p [cd] i_{(p+1)} \ldots} = \left(K_{b i_2 \ldots ; \ldots i_p} R^b{}_{i_1 cd} + \ldots + K_{i_1 \ldots ; \ldots i_{(p-1)} b} R^b{}_{i_p cd} \right)_{; i_{(p+1)} \ldots} \tag{5}$$

to have

$$
\begin{aligned}
K_{i_1\ldots i_r;i_{(r+1)}\ldots i_{(r+s+1)}} &= K_{i_1\ldots i_r;(i_{(r+1)}\ldots i_{(r+s+1)})} \\
&\quad + \text{antisymmetrized derivatives of } \; \mathbf{K} \qquad (6) \\
&= K^{(r)(s+1)}_{i_1\ldots i_{(r+s+1)}} + \\
&\quad \text{expressions in } \; \mathbf{K}^{(r)(0)},\ldots,\mathbf{K}^{(r)(s-1)}. \qquad (7)
\end{aligned}
$$

In (6) antisymmetrized derivatives of $\mathbf{K}$ of order $s+1$ have been reduced to derivatives of order $s-1$ and each of them can be split in turn into a fully symmetrized $s-1$ order derivative (a component of $\mathbf{K}^{(r)(s-1)}$) and antisymmetrized derivatives of order $s-3$ and so on using the RICCI identity.

The first step is completed by a symmetrization over the indices $i_{(r+1)}\ldots i_{(r+s)}$ giving

$$
\begin{aligned}
K_{i_1\ldots i_r;(i_{(r+1)}\ldots i_{(r+s)})i_{(r+s+1)}} &= K^{(r)(s)}_{i_1\ldots i_{(r+s)};i_{(r+s+1)}} \\
&= K^{(r)(s+1)}_{i_1\ldots i_{(r+s+1)}} + \\
&\quad \text{expressions in } \; \mathbf{K}^{(r)(0)},\ldots,\mathbf{K}^{(r)(s-1)}. \qquad (8)
\end{aligned}
$$

i.e. equations of the form

$$\mathbf{D} \;\; \mathbf{K}^{(r)(s)} \;\; \simeq \;\; \mathbf{K}^{(r)(s+1)}, \;\; s = 0, 1, \ldots. \qquad (9)$$

Although the precise form of the right hand side (rhs) of (9) for arbitrary s is not given it can be easily calculated with the following procedure.

A tensor W with indices $i_1 \ldots i_p$ is split into its symmetric part and antisymmetric parts by

$$
\begin{aligned}
W_{i_1\ldots i_p} &= \frac{1}{p!}(W_{i_1\ldots i_p} + \text{all permutations of } \; \{i_1\ldots i_p\}) \\
&\quad + \frac{1}{p!}(\;(p!-1)W_{i_1\ldots i_p} - \text{all permutations of } \; \{i_1\ldots i_p\}) \qquad (10) \\
&= W_{(i_1\ldots i_p)} + \text{`second term of (10)'}. \qquad (11)
\end{aligned}
$$

Each of the $(p!-1)$ terms $W_{i_1\ldots i_p}$ in the second term on the rhs of (10) can be converted into any one of the $(p!-1)$ permutations to cancel it. This is done by repeatedly interchanging two neighboring indices such that the second bracket can be written as a sum of terms $W_{\ldots[jk]\ldots} = (W_{\ldots jk\ldots} - W_{\ldots kj\ldots})/2$. For example, with $p = 3$, a possible decomposition is

$$W_{abc} = W_{(abc)} + \frac{2}{3}W_{a[bc]} + \frac{2}{3}W_{b[ac]} + \frac{1}{3}W_{[ac]b} + \frac{1}{3}W_{[bc]a} + W_{[ab]c} \qquad (12)$$

(which is not unique due to the Jacobi identity $W_{[a[bc]]} + W_{[b[ca]]} + W_{[c[ab]]} \equiv 0$). The decomposition of $W_{(ab)c}$ as required in (8) would be identical to the rhs of (12) without the last term $W_{[ab]c}$.

This completes the first step. The system (9) has the form of structural equations (2) but so far with an unlimited number of functions and equations as s is not bounded.

In the second step the Killing equations and their differential consequences are combined with (9). The Killing equations read

$$\begin{aligned} 0 &= K_{(i_1 \ldots i_r ; i_{(r+1)})} \\ &= K^{(r)(1)}_{(i_1 \ldots i_{(r+1)})} \end{aligned} \tag{13}$$

and successive covariant differentiations and substitutions of first order derivatives of $\mathbf{K}^{(r)(s)}$ using (9) provides relations of the form

$$0 \simeq K^{(r)(s)}_{(i_1 \ldots i_{(r+1)}) i_{(r+2)} \ldots i_{(r+s)}}, \quad s > 1. \tag{14}$$

Another identity needed below is

$$\begin{aligned} S_{a_1 \ldots a_m} S_{b_1 \ldots b_{(r+1-m)}} K_{a_1 \ldots a_m b_1 \ldots b_{(r-m)} ; b_{(r+1-m)}} &= \\ -\frac{m}{(r+1-m)} \quad S_{a_1 \ldots a_m} S_{b_1 \ldots b_{(r+1-m)}} K_{a_1 \ldots a_{(m-1)} b_1 \ldots b_{(r+1-m)} ; a_m}, & \end{aligned} \tag{15}$$

for $m = 0, \ldots, r$. This results from the Killing equations through

$$\begin{aligned} 0 &= (r+1)\, K_{(a_1 \ldots a_m b_1 \ldots b_{(r-m)} ; b_{(r+1-m)})} \\ &= S_{a_1 \ldots a_m} S_{b_1 \ldots b_{(r+1-m)}} (r+1)\, K_{(a_1 \ldots a_m b_1 \ldots b_{(r-m)} ; b_{(r+1-m)})} \\ &= S_{a_1 \ldots a_m} S_{b_1 \ldots b_{(r+1-m)}} \big(K_{a_1 \ldots a_m b_1 \ldots b_{(r-m)} ; b_{(r+1-m)}} + \\ &\qquad K_{b_{(r+1-m)} a_1 \ldots a_m b_1 \ldots b_{(r-m-1)} ; b_{(r-m)}} + \\ &\qquad \ldots \\ &\qquad K_{a_2 \ldots a_m b_1 \ldots b_{(r-m)} b_{(r+1-m)} ; a1} \big) \\ &= S_{a_1 \ldots a_m} S_{b_1 \ldots b_{(r+1-m)}} \big(\, (r+1-m)\, K_{a_1 \ldots a_m b_1 \ldots b_{(r-m)} ; b_{(r+1-m)}} + \\ &\qquad m\, K_{a_1 \ldots a_{(m-1)} b_1 \ldots b_{(r+1-m)} ; a_m} \,\big) \end{aligned} \tag{16}$$

and from (16) which is valid for $m = 0, \ldots, r+1$, further (15). Identity (15) is used in the remainder of the second step to show that

$$K^{(r)(r)}_{a_1 \ldots a_r b_1 \ldots b_r} \simeq (-1)^r\, K^{(r)(r)}_{b_1 \ldots b_r a_1 \ldots a_r} \quad \text{and} \tag{17}$$

$$\mathbf{K}^{(r)(r+1)} \simeq 0. \tag{18}$$

This is done by using (15) and the RICCI identities repeatedly:

$$\begin{aligned} K^{(r)(r)}_{a_1 \ldots a_r b_1 \ldots b_r} &= S_{a_1 \ldots a_r} S_{b_1 \ldots b_r} K_{a_1 \ldots a_r ; b_1 \ldots b_r} && \{\text{by def. of } \mathbf{K}^{(r)(s)}\} \\ &= -r S_{a_1 \ldots a_r} S_{b_1 \ldots b_r} K_{b_1 a_1 \ldots a_{(r-1)} ; a_r b_2 \ldots b_r} && \{\text{by (15) with } m = r\} \\ &\simeq -r S_{a_1 \ldots a_r} S_{b_1 \ldots b_r} K_{b_1 a_1 \ldots a_{(r-1)} ; b_2 \ldots b_r a_r} && \{\text{by RICCI ident. }\} \\ &= (-1)^2 \frac{r(r-1)}{1 \cdot 2} S_{a_1 \ldots a_r} S_{b_1 \ldots b_r} K_{b_1 b_2 a_1 \ldots a_{(r-2)} ; a_{(r-1)} b_3 \ldots b_r a_r} \\ &\simeq (-1)^2 \tbinom{r}{2} S_{a_1 \ldots a_r} S_{b_1 \ldots b_r} K_{b_1 b_2 a_1 \ldots a_{(r-2)} ; b_3 \ldots b_r a_{(r-1)} a_r} \end{aligned}$$

$$
\begin{aligned}
& \dots \\
&= (-1)^r \tbinom{r}{r} S_{a_1\ldots a_r} S_{b_1\ldots b_r} K_{b_1\ldots b_r;a_1\ldots a_r} \\
&= (-1)^r K^{(r)(r)}_{b_1\ldots b_r a_1\ldots a_r}.
\end{aligned}
$$

Using this result we get further for $\mathbf{K}^{(r)(r+1)}$

$$
\begin{aligned}
K^{(r)(r+1)}_{a_1\ldots a_r b_1\ldots b_{(r+1)}} &= S_{a_1\ldots a_r} S_{b_1\ldots b_{(r+1)}} K_{a_1\ldots a_r;b_1\ldots b_{(r+1)}} \\
&\simeq (-1)^r S_{a_1\ldots a_r} S_{b_1\ldots b_{(r+1)}} K_{b_1\ldots b_r;a_1\ldots a_r b_{(r+1)}} \\
&\simeq (-1)^r S_{a_1\ldots a_r} S_{b_1\ldots b_{(r+1)}} K_{b_1\ldots b_r;b_{(r+1)} a_1\ldots a_r} \\
&= 0.
\end{aligned}
$$

The possibility to express $\mathbf{K}^{(r)(r+1)}$ in terms of $\mathbf{K}^{(r)(0)}, \ldots, \mathbf{K}^{(r)(r-1)}$ makes the structural system (9) to be finite. To summarize:

Killing equations (1) for Killing tensors $\mathbf{K}$ *of rank* r *are equivalent to structural equations* $(F_A)_{;k} = \sum_B \Gamma_{kAB} F_B$ *with* F_A *being the components* $K^{(r)(s)}_{i_1\ldots i_{(r+s)}}$ *of tensors* $\mathbf{K}^{(r)(s)}$, $0 \le s \le r$ *and* $\mathbf{K} = \mathbf{K}^{(r)(0)}$:

$$
\mathbf{D}\ \mathbf{K}^{(r)(s)} \simeq (1 - \delta^s_r)\, \mathbf{K}^{(r)(s+1)}, \quad s = 0, \ldots, r \tag{19}
$$

together with the algebraic relations (13) and (14):

$$
0 = S_{i_1\ldots i_{(r+1)}} K^{(r)(1)}_{i_1\ldots i_{(r+1)}} \tag{20}
$$

$$
0 \simeq S_{i_1\ldots i_{(r+1)}} K^{(r)(s)}_{i_1\ldots i_{(r+1)} i_{(r+2)}\ldots i_{(r+s)}}, \quad 2 \le s \le r \tag{21}
$$

and symmetries

$$
K^{(r)(s)}_{i_1\ldots i_{(r+s)}} = S_{i_1\ldots i_r} S_{i_{(r+1)}\ldots i_{(r+s)}} K^{(r)(s)}_{i_1\ldots i_{(r+s)}}, \quad 0 \le s \le r. \tag{22}
$$

The exact form of (19) and (21) has to be worked out for each individual r by using the RICCI identity (5) in the derivation of these equations whenever covariant derivatives are interchanged (see appendix A. for $r = 1, 2, 3$). The equations (20) - (22) and (17) can be used to simplify the rhs of (19).

5 Flat space

Although it has been proven in [20] that all KT's of flat space with rank ≥ 2 are fully reducible we give the explicit solution to demonstrate how easily these structural equations can be integrated.

In the special case of flat space all "$\simeq$" signs in (19),(13),(14) can be replaced by "$=$" signs and the equations (19) can be integrated in the order $s = r, \ldots, 0$ directly to obtain

$$
K^{(r)(r)}_{i_1\ldots i_{(2r)}} = L^{(r)(r)}_{i_1\ldots i_{(2r)}}
$$

$$
\begin{aligned}
K^{(r)(r-1)}_{i_1\dots i_{(2r-1)}} &= L^{(r)(r)}_{i_1\dots i_{(2r)}} x^{i_{(2r)}} + L^{(r)(r-1)}_{i_1\dots i_{(2r-1)}} \\
&\dots \\
K^{(r)(0)}_{i_1\dots i_r} &= \sum_{s=0}^{r} \frac{1}{s!} L^{(r)(s)}_{i_1\dots i_r i_{(r+1)}\dots i_{(r+s)}} x^{i_{(r+1)}} \dots x^{i_{(r+s)}}
\end{aligned}
$$

where the $L^{(r)(s)}_{i_1\dots i_{(r+s)}},\ \ s = 0,\dots,r$ are constants of integration which satisfy the symmetry conditions

$$
\begin{aligned}
L^{(r)(s)}_{i_1\dots i_r i_{(r+1)}\dots i_{(r+s)}} &= L^{(r)(s)}_{(i_1\dots i_r)(i_{(r+1)}\dots i_{(r+s)})} &(23)\\
L^{(r)(s)}_{(i_1\dots i_r i_{(r+1)})i_{(r+2)}\dots i_{(r+s)}} &= 0. &(24)
\end{aligned}
$$

In the case of Killing vectors ($r = 1$) in space-time ($n = 4,\ i_k = 0,\dots 3$), the $L^{(1)(0)}_{i_1}$ correspond to boosts, the $L^{(1)(1)}_{\alpha\beta},\ \ \alpha,\beta = 1,2,3$ correspond to Galilei rotations and the $L^{(1)(1)}_{0\alpha}$ to Lorentz rotations.

The example of flat space suggests that, if all integrability conditions are fulfilled then if possible, to integrate the structural equations first and only afterwards to solve the algebraic equations as conditions for the constants of integration.

6 Integrability conditions

Integrability conditions for the structural equations are obtained by calculating $[\mathbf{DD}]\,\mathbf{K}^{(r)(r)}$ from (19) and again by using the RICCI identity. The resulting identity is a linear algebraic condition for the $\mathbf{K}^{(r)(0)}\dots\mathbf{K}^{(r)(r)}$ with coefficients depending on the Riemann tensor and its derivatives of order up to r. The coefficients of the $\mathbf{K}^{(r)(s)}$ vanish in spaces of constant curvature. Further integrability conditions can be obtained through differentiation and substitution of derivatives of $\mathbf{K}^{(r)(s)}$ through the structural equations to obtain more equations algebraic in $\mathbf{K}^{(r)(s)}$. This procedure can be repeated until such a differentiation and substitution step would only generate equations that are functionally dependent on the equations already known. Then neither would further steps generate new conditions as shown in [21].

In general, the integrability conditions resulting from the computations above will be algebraic expressions in all the $\mathbf{K}^{(r)(0)},\dots,\mathbf{K}^{(r)(r)}$ and will involve many very long equations. For computational purposes it would be interesting to know whether linear combinations of these equations always exist, that involve only a subset of the unknowns, for example no components $\mathbf{K}^{(r)(r)}$. Such, probably shorter, relations could be used first in an attempt to solve the structural equations.

To delete $\mathbf{K}^{(r)(r)}$ from the integrability conditions one could use relations (17) and (21) for $s = r$:

- As the integrability conditions are derived from $K^{(r)(r)}_{i_1,\dots,i_{(2r)};[i_{(2r+1)}i_{(2r+2)}]}$, a symmetrization of the integrability conditions wrt. $S_{i_1,\ldots,\imath_{(r+1)}}$ will yield relations equivalent to

$$(S_{i_1,\ldots,\imath_{(r+1)}} K^{(r)(r)}_{i_1\ldots i_{(2r)}})_{;[i_{(2r+1)} i_{(2r+2)}]}$$

which will not involve $\mathbf{K}^{(r)(r)}$ due to the symmetry (21)

$$S_{i_1,\ldots,\imath_{(r+1)}} K^{(r)(r)}_{i_1\ldots i_{(2r)}} \simeq 0.$$

- A similar idea is to apply (17) and to calculate

$$(K^{(r)(r)}_{i_1\ldots i_{(2r)}} - (-1)^r K^{(r)(r)}_{i_{(2r)}\ldots i_1})_{;[i_{(2r+1)} i_{(2r+2)}]},$$

  i.e. to calculate the difference between the integrability conditions and $(-1)^r$ times the integrability conditions with indices $i_1, \ldots, i_{2r}$ reversed.

The reason that this approach does not work and that only identities result this way, is that the integrability conditions for the structural equations $\mathbf{D}\,\mathbf{K}^{(r)(s)} = \ldots, s < r$, are the equations $\mathbf{D}\,\mathbf{K}^{(r)(s+1)} = \ldots$ and are therefore identically satisfied. Only the equations $\mathbf{D}\,\mathbf{K}^{(r)(r)} = \ldots$ have non-trivial integrability conditions but they involve $\mathbf{K}^{(r)(r)}$. A computer calculation for rank 2 confirms that only identities result.

Therefore, the only way to get length-reduced integrability conditions is to compute contractions of them not aiming at eliminating $\mathbf{K}^{(r)(r)}$.

7 The combined system of structural equations and integrability conditions

In order to solve the combined system of structural equations and integrability conditions, any single algebraic condition can be solved for one of the F_A and this F_A can be substitued in all other algebraic conditions and in the structural equations. The n structural equations with a first order derivative of F_A will become either identities 0=0 or new algebraic conditions after derivatives of other F_B resulting through this substitution have been replaced by other structural equations. This process is repeated until there is no algebraic condition left. If no unevaluated F_A is left then all F_A are zero and no KT of rank r exists. Otherwise the remaining equations still form a system of structural equations with all integrability conditions satisfied. If this involves k functions F_A then these are $n \times k$ equations with the general solution having k constants, i.e. this space has k KTs of rank r.

8 Non reducible Killing tensors and the rigidly rotating disk of dust

In order to find the number of KT's of rank r one can compute integrability conditions and substitute them in structural equations as described above. Another way, enabling numerical computations locally is the following.

With $Z_n^r = \frac{1}{r+1}\binom{n+r-1}{r}\binom{n+r}{r}$ as the maximal number of KT's of rank r ([20]) we can obtain the number of KT's of a given space by subtracting the number of functionally independent algebraic integrability conditions from Z_n^r. The result is the total number of reducible and non-reducible rank r KT's. If one knows all KV's and KT's of rank less than r (including the metric as a trivial KT of rank 2) then it is no problem to find the number of reducible KT's of rank r by generating all linear independent fully symmetrized products of lower rank KT's that give a tensor of rank r. The number of non-reducible KT's is therefore the difference of both numbers. To complete this counting we needed the lower rank KT's explicitly. It is not enough to know only the number of non-reducible lower rank KT's. For example, in 4-dimensional flat space the 10 KV's provide 10(10+1)/2=55 symmetrized products, but only 50 of them are linearly independent which is the maximal number of KT's of rank 2.

To find only the number of non-trivial Killing tensors one can do the calculation in a single point because the integrability conditions are algebraic in the $\mathbf{K}^{(r)(s)}$. This can become important like in the following problem.

In a number of publications ([12]-[17]), Meinel and Neugebauer gave the exact solution of Einstein's Field Equations describing the rigidly rotating disk of dust.

In [15] it is shown that for fixed angular momentum there exists an upper limit for the total mass of the disk and at this limit a transition to the Kerr solution occurs which is known to have a rank 2 Killing tensor.

Another limiting procedure leads to the cosmological solution ([17])

$$ds^2 = e^{2U}\left[e^{2k}(dr^2 + r^2 d\vartheta^2) + r^2 \sin^2\vartheta\, d\varphi^2\right] - e^{2U}(dt + a d\phi)^2$$

$$e^{2U} = \Omega^2 r^2 \cdot \frac{\cos^4\vartheta + 6\cos^2\vartheta - 3}{\cos^2\vartheta + 1},$$

$$a = \frac{2}{\Omega^2 r} \cdot \frac{\cos^2\vartheta - 1}{\cos^4\vartheta + 6\cos^2\vartheta - 3}$$

$$e^{2k} = \frac{1}{4}(\cos^4\vartheta + 6\cos^2\vartheta - 3).$$

Applying `CLASSYM` [26], 4 Killing vectors are found, i.e. two more than the dust disk has:

$$\partial_t, \qquad \partial_\phi, \qquad \left(\frac{t^2}{2} + \frac{1}{8r^2\Omega^2}\right)\partial_t - rt\partial_r - \frac{1}{2r\Omega}\partial_\phi, \qquad t\partial_t - r\partial_r.$$

Also in the Newtonian spherical symmetric limit, the Kepler potential, there is an additional vectorial conservation law quadratic in the momentum - the Runge-Lenz vector. It is the classical analog of a rank 2 Killing tensor.

Motivated through these extra conservation laws the question arises whether the dust disk has a rank 2 Killing tensor as well. Although the metric is known analytically, a direct investigation would be very difficult as it involves hyperelliptic integrals.

An alternative is:

- Compute the numerical value of the Riemann tensor and its first and second covariant derivatives to a high precision in a single point which is neither on the symmetry axis nor on the plane $z = 0$.
- Compute the coefficients of the 10 components $K_{ij}^{(2)(0)}$, the 20 $K_{ijk}^{(2)(1)}$ satisfying (20) and 100 $K_{ijkl}^{(2)(2)}$ satisfying (22) in the system of symmetry conditions (21) (or (29) below) and all integrability conditions (30).
- Compute the rank of the coefficient matrix of the 130 unknowns in these equations and to subtract it from 130, to find an upper bound for the number of Killing tensors.
- Subtract 4 from this number for the metric tensor and the 3 symmetrized products of the two Killing vectors in order to obtain an upper bound for the number of non-trivial KTs.
- If that number is non-zero then compute numerical values of third order derivatives of the Riemann tensor and differentiate the integrability conditions to compute the rank of the coefficient matrix of the increased set of equations. Repeat this process until the rank does not change or the number of non-trivial KTs is zero.

To perform this procedure for the dust disk, numerical values for the Riemann tensor and its first and second order derivatives had been provided by A. Kleinwächter [11] with an accuracy of 15 positions. After the first cycle at least a further cycle would have been necessary but the first set of integrability conditions was already too big to be differentiated again and the $\mathbf{D}\ \mathbf{K}^{(r)(s)}$ to be substituted in a 24MB `REDUCE` session. Also, highly accurate numerical values for all third order derivatives of the Riemann tensor would have become necessary.

An alternative test is to check the Newtonian rigidly rotating dust disk for conservation laws quadratic in the momentum. The Hamiltonian describing motion of a test particle in this potential is ([17])

$$H = \frac{1}{2} G_{ij} p^i p^j + V,$$

$$G_{ij} = \frac{1}{mr^2} \mathrm{diag} \left(\frac{1+\xi^2}{\xi^2+\eta^2},\ \frac{1-\eta^2}{\xi^2+\eta^2},\ \frac{1}{(1+\xi^2)(1-\eta^2)} \right),$$

$$V = -\frac{\omega^2 r^2}{\pi} \left(\frac{4}{3} \arctan \xi + \left(\xi - (\xi^2 + \frac{1}{3}) \arctan \xi \right) (1 - 3\eta^2) \right).$$

In order to have conservation laws quadratic in p^i

$$\text{const.} = K_{ij}p^i p^j + L,$$

the symmetric coefficient matrix K_{ij} has to satisfy the Killing tensor equation as a necessary condition

$$K_{(ij;m)} = 0. \tag{25}$$

The covariant derivative is computed with G_{ij}. Using `CLASSYM` with the command `(dimension 3)`, 10 equations (25) for 6 unknowns K_{ij} are generated. The general solution of this system found by `CRACK` is

$$K_{ij} = c_1 G_{ij} + c_2 \delta_i^{\varphi} \delta_j^{\varphi} \frac{1}{(1-\eta^2)^2(1+\xi^2)^2}$$

with the second Killing tensor being the square of the Killing vector ∂_ϕ .

As the Newtonian dust disk does not have any non-trivial conservation laws quadratic in the momentum, it is highly unlikely, that the relativistic case has any non-trivial rank 2 Killing tensor.

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Appendix A: Structural equations for Killing tensors of rank 1,2 and 3

The structural equations and integrability conditions provided below have been obtained with the help of a special purpose REDUCE computer algebra program [9], that performs tensor operations as described in section 3. In addition, it is able to factor out symmetry and antisymmetry operators S, A as is seen below in order to compactify expressions. The first upper index indicating the rank of $\mathbf{K}$ has been omitted as well as the ( ) of the second upper index to improve readability.

Rank one

For rank $r = 1$ the structural equations take the well-known form

$$K^0_{a;b} = K^1_{ab} \tag{26}$$

$$K^1_{ab;c} = K^0_m R^m{}_{cba} \tag{27}$$

with integrability conditions

$$0 = A_{ab}\, A_{cd}\, (\, K^0_m R^m{}_{cba;d} +\, K^1_{mc} R^m{}_{dab} +\, K^1_{ma} R^m{}_{bcd}) \tag{28}$$

where A is the antisymmetrization operator. The algebraic condition (13) takes the form

$$0 = S_{ab}\;\, K^1_{ab}.$$

Rank two

For rank $r = 2$ the structural equations take the form

$$\begin{aligned}
K^0_{ab;c} &= K^1_{abc} \\
K^1_{abc;d} &= K^2_{abcd} + S_{ab}\,(\, K^0_{am} R^m{}_{bcd}) \\
K^2_{abcd;e} &= 2\, S_{ab}\,(\, S_{cde}\,(\, K^0_{cm} R^m{}_{bda;e} -\, K^0_{cm} R^m{}_{dea;b} + \\
&\qquad 2\, K^1_{acm} R^m{}_{deb} -\, K^1_{cdm} R^m{}_{bea}) \\
&\qquad + S_{cd}\,(\, \tfrac{1}{3}\, K^0_{am} R^m{}_{cdb;e} + \tfrac{2}{3}\, K^0_{am} R^m{}_{ecb;d} - \\
&\qquad \tfrac{1}{3}\, K^1_{abm} R^m{}_{dec} + 2\, K^1_{amc} R^m{}_{edb} + \\
&\qquad K^1_{ame} R^m{}_{cdb}\,)\,)
\end{aligned}$$

and algebraic conditions (13) (14) (17) become

$$\begin{aligned}
0 &= S_{abc}\;\; K^1_{abc} \\
0 &= S_{abc}\,(\, K^2_{abcd} +\, K^0_{am} R^m{}_{bcd})
\end{aligned}$$

$$K^2_{abcd} -\, K^2_{cdab} = 2\, S_{ab}\, S_{cd}\,(\, K^0_{am} R^m{}_{cdb} -\, K^0_{cm} R^m{}_{abd}\,) \tag{29}$$

The integrability conditions are

$$\begin{aligned}
0 =\;& S_{ab}\, S_{cd}\, A_{ef}\, \big[K^0_{am}(2R^m{}_{ecb;df} - 3/2 R^m{}_{nfe} R^n{}_{cdb} - R^m{}_{nfd} R^n{}_{ceb} - R^m{}_{bnc} R^n{}_{efd} \\
& + R^m{}_{nfb} R^n{}_{dec} + R^m{}_{cnd} R^n{}_{efb} - 4R^m{}_{nfd} R^n{}_{ecb} - 2R^m{}_{ndb} R^n{}_{efc} + R^m{}_{fnd} R^n{}_{ceb} \\
& + R^m{}_{fnd} R^n{}_{ecb} - R^m{}_{fnb} R^n{}_{dec}) + K^0_{cm}(R^m{}_{bea;df} - R^m{}_{dea;bf} - R^m{}_{eda;bf} \\
& + R^m{}_{nfe} R^n{}_{bda} + R^m{}_{nfd} R^n{}_{bea} - R^m{}_{anb} R^n{}_{efd} - R^m{}_{nfb} R^n{}_{dea} + R^m{}_{and} R^n{}_{efb} \\
& + 2R^m{}_{nfb} R^n{}_{eda} + 2R^m{}_{ndb} R^n{}_{efa} - R^m{}_{fnd} R^n{}_{bea} + R^m{}_{fnb} R^n{}_{dea} + 4R^m{}_{fnb} R^n{}_{eda} \\
& + 3R^m{}_{dnb} R^n{}_{efa}) + K^0_{em}(R^m{}_{bca;df} - R^m{}_{cda;bf} + R^m{}_{nfd} R^n{}_{bca} - R^m{}_{nfb} R^n{}_{cda} \\
& - 3R^m{}_{ndb} R^n{}_{fca} - R^m{}_{fnd} R^n{}_{bca} + R^m{}_{fnb} R^n{}_{cda} - 3R^m{}_{dnb} R^n{}_{fca}) - K^1_{abm} R^m{}_{dec;f} \\
& + 2K^1_{acm} R^m{}_{deb;f} + 2K^1_{acm} R^m{}_{edb;f} + 2K^1_{aem} R^m{}_{cdb;f} + 6K^1_{amc} R^m{}_{edb;f} \\
& + 2K^1_{ame} R^m{}_{cdb;f} - 2K^1_{ame} R^m{}_{fcb;d} - K^1_{cdm} R^m{}_{bea;f} - 2K^1_{cem} R^m{}_{bda;f} \\
& - K^1_{cme} R^m{}_{bda;f} - K^1_{cme} R^m{}_{bfa;d} + K^1_{cme} R^m{}_{dfa;b} + K^1_{cme} R^m{}_{fda;b}
\end{aligned}$$

$$+K^1_{emf}R^m{}_{bca;d} - K^1_{emf}R^m{}_{cda;b} + 3K^2_{abcm}R^m{}_{efd} + K^2_{abem}R^m{}_{dfc}$$
$$-6K^2_{acdm}R^m{}_{efb} - 2K^2_{acem}R^m{}_{dfb} + 4K^2_{acem}R^m{}_{fdb} + 6K^2_{aecm}R^m{}_{fdb}$$
$$+2K^2_{aefm}R^m{}_{cdb} + K^2_{cdem}R^m{}_{bfa} - 2K^2_{cefm}R^m{}_{bda} \Big] \tag{30}$$

Rank three

For rank $r = 3$ the structural equations take the form

$$\begin{aligned}
K^0_{abc;d} &= K^1_{abcd} \\
K^1_{abcd;e} &= K^2_{abcde} + S_{abc}\, A_{de}\, 3K^0_{abm}R^m{}_{edc} \\
K^2_{abcde;f} &= K^3_{abcdef} - S_{abc}\, S_{de}\, (\, K^0_{abm}R^m{}_{cfd;e} + 2/3K^1_{abcm}R^m{}_{efd} \\
&\qquad + 3K^1_{abmd}R^m{}_{cfe}) \\
K^3_{abcdef;g} &= S_{abc}\, S_{def}\, (\text{ about 200 terms involving } K^0, K^1, K^2)
\end{aligned}$$

Symmetries of $\mathbf{K}^1, \ldots, \mathbf{K}^3$ as a consequence of the Killing equations (13) and their derivatives (14) are

$$\begin{aligned}
0 &= S_{abcd}\;\; K^1_{abcd} \\
0 &= S_{abcd}\,(\, 2K^2_{abcde} - 3K^0_{abm}R^m{}_{dec}) \\
0 &= S_{abcd}\,(\, 2K^3_{abcdef} - K^0_{abm}\left(3R^m{}_{dec;f} + R^m{}_{dfc;e} + R^m{}_{cfe;d}\right) \\
&\qquad -2/3K^1_{abcm}\left(R^m{}_{efd} + R^m{}_{dfe}\right) \\
&\qquad -3K^1_{abmc}R^m{}_{dfe} - 3K^1_{abme}R^m{}_{dfc} - 3K^1_{abmf}R^m{}_{dec})
\end{aligned}$$

Integrability conditions for rank r=3 are very large in size which may be discouraging. But products vanish as soon as only one factor vanishes and for a given metric with many vanishing components of the curvature tensor expressions might shrink drastically. Also, for a given metric to be investigated it would be enough to compute only one integrability condition and to use that to reduce the system of structural equations as described in section 7. Another simplification is given in dimensions lower than 4. In three dimensions the Riemann tensor can be expressed through the Ricci tensor and in two dimensions through the Ricci scalar.

Appendix B: Reducibility of KT's of flat space

In this appendix we prove that the general rank r Killing tensor of flat space

$$K_{a_1\ldots a_r} = \sum_{s=0}^{r} \frac{1}{s!}\, L^{(r)(s)}_{a_1\ldots a_r b_1\ldots b_s} x^{b_1} \ldots x^{b_s} \tag{31}$$

is fully reducible, i.e. it can be written as a sum of fully symmetrized products of Killing vectors.

Although this reducibility was already proven before in [20], we included this direct version for two reasons.

- As a byproduct we will get a general one-to-one correspondence between any tensors $L_{a_1\ldots a_r b_1\ldots b_s}$ and $C_{a_1\ldots a_r b_1\ldots b_s}$ both of rank $r+s$, $r \geq s$ where $\mathbf{L}$ has symmetries $L_{a_1\ldots a_r b_1\ldots b_s} = L_{(a_1\ldots a_r)(b_1\ldots b_s)}$, $L_{(a_1\ldots a_r b_1)(b_2\ldots b_s)} = 0$ and $\mathbf{C}$ is antisymmetric in each pair a_i, b_i and is furthermore symmetric in $a_{s+1}\ldots a_r$ and symmetric w.r.t. exchanges of pairs $a_i b_i \leftrightarrow a_k b_k$.
- In the non-flat case an analogous calculation, starting from (34) but only using $\mathbf{K}^{(r)(s)}$ instead of $\mathbf{L}^{(r)(s)}$ and the RICCI-identity whenever covariant derivatives are interchanged, shows an equivalence between tensors $\mathbf{K}^{(r)(s)}$ and $\mathbf{C}^{(r)(s)}$.

  In the work of Hauser & Malhiot [6, 7] structural equations for rank 2 Killing tensors are formulated using tensors $\mathbf{L}, \mathbf{M}$. These tensors are identical to $\mathbf{C}^{(2)(1)}$, $\mathbf{C}^{(2)(2)}$ apart from a constant factor and an interchange of indices $C^1_{abc} \leftrightarrow L_{acb}$, $\quad C^2_{abcd} \leftrightarrow M_{acbd}$. As the constructive method to express $\mathbf{C}^{(r)(s)}$, $\mathbf{K}^{(r)(s)}$ in terms of each other does work for arbitrary rank, we achieved as well a generalization of the formalism of Hauser & Malhiot [6, 7] to arbitrary dimension. This formulation of structural equations using L, M was prefered by Hauser & Malhiot when a null tetrad formulation is to be used to formulate the structural equations for a given spacetime.

Proof: An n-dimensional flat space has the Killing vectors ξ^A, $\ A = 1, \ldots, n$ with components $\xi^A_i = \delta^A_i$ and the Killing vectors ζ^α, $\ \alpha = 1, \ldots, n(n-1)/2$ with components $\zeta^\alpha_i = \epsilon^\alpha_{ij} x^j$, $\ \epsilon^\alpha_{ij} = -\epsilon^\alpha_{ji} \in (-1, 0, 1)$. To generate the term $L^{(r)(s)}_{a_1\ldots a_r b_1\ldots b_s}\ x^{b_1}\ldots x^{b_s}$ in (31) from these Killing vectors, we have to consider a linear combination of products symmetrized over the indices $a_1, \ldots, a_r$ of s vectors ζ^α (due to the power s of x^i) and $r-s$ vectors ξ^A:

$$S_{a_1\ldots a_r}\, C^{(r)(s)}_{\alpha_1\ldots\alpha_s A_1\ldots A_{(r-s)}} \epsilon^{\alpha_1}_{a_1 b_1} x^{b_1} \ldots \epsilon^{\alpha_s}_{a_s b_s} x^{b_s} \delta^{A_1}_{a_{(s+1)}} \ldots \delta^{A_{(r-s)}}_{a_r} \tag{32}$$

where summation over identical Latin indices goes from 1 to n and summation over Greek indices goes from 1 to $n(n-1)/2$. We have to show that coefficients $\mathbf{C}^{(r)(s)}$ can be chosen such that any $\mathbf{L}^{(r)(s)}$ with its symmetries (23) can be expressed through

$$L^{(r)(s)}_{a_1\ldots a_r b_1\ldots b_s} = S_{a_1\ldots a_r} S_{b_1\ldots b_s}\, C^{(r)(s)}_{\alpha_1\ldots\alpha_s a_{(s+1)}\ldots a_r} \epsilon^{\alpha_1}_{a_1 b_1} \ldots \epsilon^{\alpha_s}_{a_s b_s}. \tag{33}$$

with $S_{b_1\ldots b_s}$ taking over the symmetrizing effect of $x^{b_1}\ldots x^{b_s}$ or, equivalently,

$$L^{(r)(s)}_{a_1\ldots a_r b_1\ldots b_s} = S_{a_1\ldots a_r} S_{b_1\ldots b_s}\, C^{(r)(s)}_{a_1\ldots a_r b_1\ldots b_s}\ ,\ 0 \leq s \leq r \tag{34}$$

with coefficients $C^{(r)(s)}_{a_1\ldots a_r b_1\ldots b_s}$ that are antisymmetric in each pair $a_m b_m$, $\ 1 \leq m \leq s$.

The symmetry (24) of $\mathbf{L}^{(r)(s)}$, i.e. the vanishing of $\mathbf{L}^{(r)(s)}$ symmetrized over the first $r+1$ indices is also satisfied by the rhs of (34). Such a symmetrization would involve for each term of the sum given by the rhs of (34) at least one pair of indices $a_m b_m$ and would therefore vanish.

In the following we will give a constructive proof of the existence of $\mathbf{C}^{(r)(s)}$ by showing how to calculate them from given $\mathbf{L}^{(r)(s)}$. A pair of indices p_iq_i will stand for a pair a_ib_i with either $p_i = a_i$, $q_i = b_i$ or $p_i = b_i$, $q_i = a_i$. The calculation is done by a sequence of antisymmetrizations successively over the index pairs $a_1b_1, a_2b_2, \ldots, a_sb_s$. A single antisymmetrization step is done through manipulating each term by the following step

$$
\begin{aligned}
& S_{a_1\ldots a_r}S_{b_1\ldots b_s}\,L^{(r)(s)}_{p_1\ldots p_{(i-1)}a_ip_{(i+1)}\ldots p_rq_1\ldots q_{(i-1)}b_iq_{(i+1)}\ldots q_s} \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}\,L^{(r)(s)}_{p_1\ldots p_{(i-1)}a_ip_{(i+1)}\ldots p_rb_iq_1\ldots q_{(i-1)}q_{(i+1)}\ldots q_s} \\
& \text{(It is assumed that among the first } r \text{ indices of } \mathbf{L}^{(r)(s)} \text{ there} \\
& \text{are } m\ \ a\text{-indices and } (r-m)\ \ b\text{-indices. The} \\
& \text{identity (16) then gives:)} \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}(L^{(r)(s)}_{p_1\ldots p_{(i-1)}a_ip_{(i+1)}\ldots p_rb_iq_1\ldots q_{(i-1)}q_{(i+1)}\ldots q_s} - \\
& \frac{1}{(r+1)}((r+1-m)\,L^{(r)(s)}_{p_1\ldots p_{(i-1)}a_ip_{(i+1)}\ldots p_rb_iq_1\ldots q_{(i-1)}q_{(i+1)}\ldots q_s} + \\
& \qquad m\,L^{(r)(s)}_{p_1\ldots p_{(i-1)}b_ip_{(i+1)}\ldots p_ra_iq_1\ldots q_{(i-1)}q_{(i+1)}\ldots q_s})) \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}\frac{m}{(r+1)}(L^{(r)(s)}_{p_1\ldots p_{(i-1)}a_ip_{(i+1)}\ldots p_rb_iq_1\ldots q_{(i-1)}q_{(i+1)}\ldots q_s} - \\
& \qquad L^{(r)(s)}_{p_1\ldots p_{(i-1)}b_ip_{(i+1)}\ldots p_ra_iq_1\ldots q_{(i-1)}q_{(i+1)}\ldots q_s}) \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}\frac{m}{(r+1)}(L^{(r)(s)}_{p_1\ldots p_{(i-1)}a_ip_{(i+1)}\ldots p_rq_1\ldots q_{(i-1)}b_iq_{(i+1)}\ldots q_s} - \\
& \qquad L^{(r)(s)}_{p_1\ldots p_{(i-1)}b_ip_{(i+1)}\ldots p_rq_1\ldots q_{(i-1)}a_iq_{(i+1)}\ldots q_s})
\end{aligned}
$$

i.e. if m of the first r indices of $L^{(r)(s)}$ are a-indices then

$$
\begin{aligned}
& S_{a_1\ldots a_r}S_{b_1\ldots b_s}\,L^{(r)(s)}_{\ldots a_i\ldots b_i\ldots} \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}\frac{m}{(r+1)}(L^{(r)(s)}_{\ldots a_i\ldots b_i\ldots} - L^{(r)(s)}_{\ldots b_i\ldots a_i\ldots}).
\end{aligned} \tag{35}
$$

This antisymmetrization step wrt. pairs $a_ib_i,\ i = 1, \ldots, s$ is applied successively to each term that arose in the previous step when $a_{(i-1)}, b_{(i-1)}$ were antisymmetrized:

$$
\begin{aligned}
& L^{(r)(s)}_{a_1\ldots a_rb_1\ldots b_s} \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}\,L^{(r)(s)}_{a_1\ldots a_rb_1\ldots b_s} \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}\frac{r}{(r+1)}\left(L^{(r)(s)}_{a_1\ldots a_rb_1\ldots b_s} - L^{(r)(s)}_{b_1a_2\ldots a_ra_1b_2\ldots b_s}\right) \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}\frac{r}{(r+1)}\left(\frac{r}{(r+1)}\left(L^{(r)(s)}_{a_1\ldots a_rb_1\ldots b_s} - L^{(r)(s)}_{a_1b_2a_3\ldots a_rb_1a_2b_3\ldots b_s}\right)\right. \\
& \qquad \left. -\frac{(r-1)}{(r+1)}\left(L^{(r)(s)}_{b_1a_2\ldots a_ra_1b_2\ldots b_s} - L^{(r)(s)}_{b_1b_2a_3\ldots a_ra_1a_2b_3\ldots b_s}\right)\right) \\
& \ldots \\
= \; & S_{a_1\ldots a_r}S_{b_1\ldots b_s}\,C^{(r)(s)}_{a_1\ldots a_rb_1\ldots b_s}.
\end{aligned}
$$

The expressions in $\mathbf{L}^{(r)(s)}$ obtained finally are identified with $\mathbf{C}^{(r)(s)}$.

For example, for $r = 2$ the method yields

$$
\begin{aligned}
C^{(2)(1)}_{a_1 a_2 b_1} &= \frac{2}{3} \left(L^{(2)(1)}_{a_1 a_2 b_1} - L^{(2)(1)}_{b_1 a_2 a_1} \right) \\
C^{(2)(2)}_{a_1 a_2 b_1 b_2} &= \frac{2}{9} \left(2\ L^{(2)(2)}_{a_1 a_2 b_1 b_2} - 2\ L^{(2)(2)}_{a_1 b_2 b_1 a_2} \right. \\
&\qquad \left. - L^{(2)(2)}_{b_1 a_2 a_1 b_2} + L^{(2)(2)}_{b_1 b_2 a_1 a_2} \right).
\end{aligned}
$$

This algorithm for computing $\mathbf{C}^{(r)(s)}$ completes the proof of reducibility of all Killing tensors of flat space to linear combinations of symmetrized products of Killing vectors.

References

[1] Benenti, S. *Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation* J. Math. Phys., **38**, no 12, (1997) 6578-6602.

[2] Carter, B. *Global structure of the Kerr Family of Gravitational Fields* Phys. Rev. D, **174**, no 5, (1968) 1559-1571.

[3] Collinson, C. D. *Special quadratic first integrals of geodesics*, J. Phys. A: Gen. Phys., **4**, (1971), 756-760.

[4] Eisenhart, L. P. *Riemannian Geometry* (Princeton U. P. Princeton, N. J. 1966), p. 128 ff.

[5] Geroch, R. P. Commun. Math. Phys. **13** (1969), 180.

[6] Hauser, I. and Malhiot, R. J. *Structural equations for Killing tensors of order two. I* J. Math. Phys. **16**, No. 1 (1975), 150-152.

[7] Hauser, I. and Malhiot, R. J. *Structural equations for Killing tensors of order two. II* J. Math. Phys. **16**, No. 8 (1975), 1625-1629.

[8] Hauser, I. and Malhiot, R. J. *On space-time Killing tensors with a Segre characteristic [(11)(11)]* J. Math. Phys. **17**, No. 7 (1976), 1306-1312.

[9] Hearn, A. C. and Fitch, J. P. *REDUCE, User's Manual 3.6* Konrad-Zuse-Zentrum Berlin (1995).

[10] Kalnins, E. G. and Miller, W. M. *Killing tensors and variable separation for Hamilton-Jacobi and Helmholz equations* SIAM J. Math. Anal. **11** (1980), p. 1011 ff.

[11] Kleinwächter, A., private communication.

[12] Neugebauer, G. and Meinel, R., Astrophys. J. **414** (1993) L97.

[13] Neugebauer, G. and Meinel, R., Phys. Rev. Lett. **73** (1994) 2166.

[14] Neugebauer, G. and Meinel, R., Phys. Rev. Lett. **75** (1995) 3046.

[15] Neugebauer, G., Kleinwächter, A. and Meinel, R., Helv. Phys. Acta **69** (1996) 472.

[16] Nijenhuis, A., Nederl. Akad. Wetensch. Proc. Ser. A **58** (1950) 390.

[17] Meinel, R., in Proceedings of the Second Mexican School on Gravitation and Mathematical Physics, Eds. A. Garcia, C. Lämmerzahl, A. Macias, T. Matos, D. Nuñez (Preprint gr-qc/9703077).

[18] Thomas, T. Y. *The fundamental theorem on quadratic first integrals*, Proc. **32**, (1946), 10-15.

[19] Thompson, G. *Polynomial constants of motion in flat space* J. Math. Phys. **25**, No. 12 (1984), 3474-3478.

[20] Thompson, G. *Killing tensors in spaces of constant curvature* J. Math. Phys. **27**, No. 11 (1986), 2693-2699.

[21] Veblen, O. and Thomas, T. Y. *The Geometry of Paths*, Trans. Am. Math. Soc., **25**, (1923), 551-608.

[22] Weir, G. J. *Conformal Killing tensors in reducible spaces* J. Math. Phys. **18**, No. 9 (1977), 1782-1787.

[23] Wolf, T. and Brand, A. *The Computer Algebra Package* `CRACK` *for Investigating PDEs*, Manual for the package `CRACK` distributed with REDUCE.

[24] Wolf, T. and Brand, A. *Examples of the investigation of differential equations with modularized programs*, Mathl. Comput. Modelling 25 (1997), no 8/9, 133 - 139.

[25] Wolf, T., *The program CRACK for solving PDEs in General Relativity*, in F.W. Hehl, R.A. Puntigam, and H. Ruder (Eds.): Relativity and Scientific Computing: Computer Algebra, Numerics, Visualization Springer Verlag, 1996, p. 241-258.

[26] Wolf, T., Grebot, G. *Automatic symmetry investigation of space-time metrics* Int. J. of Mod. Phys. D, **3**, (1994) 323-326.

[27] Woodhouse, N. M. J. Killing Tensors and the Separation of the Hamilton-Jacobi Equation Commun. Math. Phys. **44**, No. 9 (1975).