Partial and Complete Linearization of PDEs Based on Conservation Laws

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Abstract. A method based on infinite parameter conservation laws is described to factor linear differential operators out of nonlinear partial differential equations (PDEs) or out of differential consequences of nonlinear PDEs. This includes a complete linearization to an equivalent linear PDE (-system) if that is possible. Infinite parameter conservation laws can be computed, for example, with the computer algebra package ConLaw.

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1. Introduction

With the availability of computer algebra programs for the automatic computation of all conservation laws up to a given differential order of the integrating factors (as described in [5], [6]) conservation laws have been found that involve arbitrary functions, i.e. infinitely many parameters. In this paper we show how based on such conservation laws a linear differential operator can be factored out of a combination of the original nonlinear partial differential equations (PDEs) and their differential consequences. Possible outcomes include

- a complete linearization into an equivalent linear system,
- a partial linearization in the sense that a linear differential operator is factored out, splitting the problem into a linear one plus a non-linear problem of lower order and often fewer independent variables (e.g. ordinary differential equations (ODEs)),
- the derivation of at least one linear equation from a nonlinear system (with the possibility of deriving further linear equations for the new mixed linear-nonlinear system).

An advantage of the procedure to be presented is that conservation laws need not be given explicitly in terms of the arbitrary functions. It is enough to have the conservation law determining conditions solved up to the solution of a system of consistent and necessarily linear PDEs which have arbitrary functions in their general solution.

The content of the paper is as follows. After comments are made on the computation of conservation laws in section 3, the four computational steps of factoring out linear differential operators are illustrated using the Liouville equation in section 4. Sufficient conditions for complete or partial linearizations are listed in section 5, followed by a discussion of computational aspects in section 6. A generalization involving the introduction of potentials in terms of which a linearization becomes possible is explained in section 7. In later sections 8, 9, an illustration is given of how the method works when nonlinear equations linearize to *inhomogeneous* equations or to *triangular linear* systems. Further examples where a complete or at least a partial linearization is possible are given in the appendix.

In this contribution we concentrate on computational aspects of the method and give examples for all of the above scenarios. An extension of the method discussing complete and partial linearizability through point and contact transformations will appear in a future publication [1], with numerous new examples and a comparison with other linearization methods found in the literature.

2. Notation

We follow the notation in [3] and denote the original nonlinear partial differential equations as $0 = \Delta_\alpha$, the dependent variables by $u^\beta, \quad \alpha, \beta = 1 \ldots q$ and the independent variables by $x^i, \quad i = 1 \ldots p$. In examples dealing with functions $u = u(x,t)$ or $u = u(x,y)$, partial derivatives are written as subscripts like $u_{xy} = \partial^2 u/(\partial x \partial y)$. If a formula already contains subscripts then ∂_i will be used for $\partial/\partial x^i$. The multi indices $_J, _K$ denote multiple partial derivatives like u^α_J which in our notation include u^α. With $\#J$ we denote the differential order, i.e. number of partial derivatives represented by $_J$. Total derivatives with respect to x^i will be denoted as D_i. We apply the convention that summation is performed over terms that involve two identical indices, one subscript and one superscript. For example, the divergence of a vector field P^i would be denoted as $D_i P^i$ $(\equiv \sum_i D_i P^i)$. The procedure to be presented repeatedly uses adjoint differential operators as follows. For given functions $f^A(x^i), \; A = 1..r$, let linear differential expressions H_k be defined as

$$H_k = a^J_{kA} \partial_J f^A, \quad k = 1, \ldots, s \; ,$$

with coefficients $a^J_{kA} = a^J_{kA}(x^i)$ and summation over A and the multi index J. The corresponding adjoint operators H^*_{Ak} are computed for arbitrary functions $G^k(x^i)$ by repeatedly reversing the product rule of differentiation for the sum $G^k H_k$ to get

$$G^k H_k = f^A H^*_{Ak} G^k + D_i \bar{P}^i \tag{1}$$

where

$$H^*_{Ak} G^k = (-1)^{\#J} \partial_J \left(a^J_{kA} G^k \right) . \tag{2}$$

and $\bar{P}^i$ are expressions resulting from integration by parts with respect to ∂_J in this computation.

3. Conservation Laws with Arbitrary Functions

Conservation laws can be formulated in different ways (see [6] for four different approaches to compute conservation laws). The form to be used in this paper is

$$D_i P^i = Q^\alpha \Delta_\alpha \tag{3}$$

where the components P^i of the conserved current and the so-called characteristic functions Q^α are differential expressions involving x^i, u^α_J. Other forms of conservation laws can easily be transformed into (3). One approach to find conservation laws for a given system of differential equations $0 = \Delta_\alpha$ is to specify a maximum differential order m of derivatives u^α_J on which P^i, Q^α may depend and then to solve condition (3) identically in x^i, u^α_J for the unknown functions P^i, Q^α. Due to the chain rule of differentiation in (3) the total derivatives D_i introduce extra derivatives u^α_K with $\#K = m+1 > m$, i.e. derivatives not occuring as variables in P^i, Q^α. Splitting with respect to these u^α_K results in an *overdetermined* and *linear* system of PDEs for P^i, Q^α.[1]

What is important in the context of this paper is that a differential Gröbner basis can be computed algorithmically and from it the dimension of the solution space can be determined, i.e. how many arbitrary functions of how many variables the general solution for P^i, Q^α depends on. In extending the capability of a program in solving conditions (3) by not only computing a differential Gröbner basis (for linear systems) but also integrating exact PDEs (see [8]) and splitting PDEs with respect to only explicitly occuring u^α_J (which here act as independent variables), the situation does not change qualitatively. The result is still either the explicit general solution or a linear system of unsolved PDEs

$$0 = C_k(x^i, u^\alpha_J, f^A), \qquad k = 1, \ldots, r \ , \tag{4}$$

for some functions $f^A(x^j, u^\beta_J)$ where this system is a differential Gröbner basis and allows one to determine algorithmically the size of the solution space. The functions f^A are either the P^i, Q^α themselves or are functions arising when integrating the conservation law conditions (3).

If the conservation law condition (3) is solved, i.e. P^i, Q^α are determined in terms of x^i, u^α_J, f^A_K possibly up to the solution of remaining conditions (4) then it is no problem to use a simple division algorithm to determine coefficients L^k satisfying

$$Q^\alpha \Delta_\alpha = D_i P^i + L^k C_k \tag{5}$$

identically in x^i, u^α_J, f^A_J. The coefficients L^k are necessarily free of f^A_J because (3) is linear and homogeneous in Q^α, P^i and this property is preserved in solving these

[1] Note that regarding (3) as an algebraic system for unknowns Q^α implies division through Δ_α and does therefore not produce Q^α which are regular for solutions u^α of the original system $\Delta_\alpha = 0$. For details regarding the ansatz for Q^α see [6].

conditions, so C_k are linear and homogeneous in f_J^A as well and L^k must therefore be free of f_J^A. We will call relation (5) a *conservation law identity* because it is satisfied identically in all x^i, u_J^α and f_J^A.

4. The Procedure

The individual steps of our method are shown in detail to demonstrate that all steps are algorithmic and can be performed by computer. The REDUCE package CONLAW has the algorithm implemented and performs it whenever a conservation law computation results in a solution involving arbitrary functions possibly up to the solution of a linear system (4).

Input

Input to the procedure is the conservation law identity (5)

$$Q^\alpha \Delta_\alpha = D_i P^i + L^k C_k \tag{6}$$

including expressions for all its constituents Q^α, P^i, L^k, C_k in terms of x^i, u^α, f^A.

To start the procedure the functions f^A have to depend only on the variables x^i. If they depend on u_J^α then a linearization will necessarily involve a change of variables. This case is treated in [1].

Step 1.

If all functions f^A depend exactly on all p independent variables x^i then proceed to step 2. Step 1 is concerned with the case that not all $f^A = f^A(x^i)$ depend on all x^i. To add the dependence of, say f^B on x^j one has to

- compute

  $$Z := \left(Q^\alpha \Delta_\alpha - D_i P^i - L^k C_k\right)\big|_{f^B(x^i) \to f^B(x^i, x^j)}$$

  which vanishes modulo $0 = \partial_j f^B$ and therefore must have the form

  $$Z = M^J \partial_J \left(f^B\right)$$

  with suitable coefficients M^J and summation over the multi index J,
- compute the adjoint Z_B^* as in (1),(2) to bring Z into the form

  $$Z = D_i \bar{P}^i + Z_B^* \partial_j f^B, \tag{7}$$

- rename $P^i + \bar{P}^i \to P^i$ and add a new condition $C_{r+1} = \partial_j f^B$ and multiplier $L^{r+1} = Z_B^*$ to arrive at a new version of the conservation law identity $Q^\alpha \Delta_\alpha = D_i P^i + L^k C_k$ where the function f^B depends now on x^j.

This process is repeated until all f^A depend on all x^i.

Example 1: We will illustrate the steps of the procedure with an investigation of the Liouville equation

$$0 = \Delta := u_{xy} - e^u. \tag{8}$$

Although it is not completely linearizable, we choose this equation because it involves computations in each of the first three steps.

For the Liouville equation a conservation law identity involving an arbitrary function $f(x)$ is given through

$$(f_x + fu_x)\Delta = D_x(-fe^u) + D_y(f_x u_x + fu_x^2/2), \quad (9)$$

i.e. $Q = f_x + fu_x$, $P^x = -fe^u$, $P^y = f_x u_x + fu_x^2/2$, $C_k = 0$. Adding a y-dependence to f requires to add to the right hand side of our identity (9) the terms

$$Z = -f_{xy}u_x - f_y {u_x}^2/2$$

which in adjoint form (7) read

$$Z = D_x(-f_y u_x) + (u_{xx} - {u_x}^2/2)f_y$$

giving the new conservation law identity

$$(f_x + fu_x)\Delta = D_x(-fe^u - f_y u_x) + D_y(f_x u_x + fu_x^2/2) + (u_{xx} - {u_x}^2/2)f_y. \quad (10)$$

Step 2.
As the Q^α are linear homogeneous differential expressions for the f^A one can compute adjoint operators $Q_A^{\alpha *}$ as in (1),(2) by expressing

$$Q^\alpha \Delta_\alpha = f^A Q_A^{\alpha *}\Delta_\alpha + D_i \bar{P}^i.$$

After renaming $P^i - \bar{P}^i \to P^i$ the conservation law identity takes the new form

$$f^A Q_A^{\alpha *}\Delta_\alpha = D_i P^i + L^k C_k. \quad (11)$$

In the case of the Liouville equation we partially integrate

$$(f_x + fu_x)\Delta = f(u_x - D_x)\Delta + D_x(f\Delta)$$

and get the conservation law identity

$$\begin{aligned} f(u_x - D_x)\Delta &= D_x(-fe^u - f_y u_x - f\Delta) + D_y(f_x u_x + fu_x^2/2) + (u_{xx} - {u_x}^2/2)f_y \\ &= D_x(-f_y u_x - fu_{xy}) + D_y(f_x u_x + fu_x^2/2) + (u_{xx} - {u_x}^2/2)f_y. \quad (12) \end{aligned}$$

Step 3.
Because the C_k are linear homogeneous differential expressions in the f^A we can compute the adjoint form of $L^k C_k$ as in (1),(2) by expressing

$$L^k C_k = f^A C_{Ak}^{\ *} L^k + D_i \bar{P}^i.$$

After renaming $P^i + \bar{P}^i \to P^i$ the conservation law identity takes the new form

$$f^A Q_A^{\alpha *}\Delta_\alpha = D_i P^i + f^A C_{Ak}^{\ *} L^k. \quad (13)$$

In our example partial integration gives

$$(u_{xx} - {u_x}^2/2)f_y = D_y((u_{xx} - {u_x}^2/2)f) - f(u_{xx} - {u_x}^2/2)_y$$

and substituted into (12) the new conservation law identity

$$\begin{aligned} f(u_x - D_x)\Delta \quad = \quad & D_x(-f_y u_x - fu_{xy}) + D_y\left(f_x u_x + fu_x^2/2 + f(u_{xx} - {u_x}^2/2)\right) \\ & -f(u_{xx} - {u_x}^2/2)_y \quad (14) \end{aligned}$$

after simplification.

Step 4.
This step does not involve any computation, it merely completes the constructive proof how linearizations are achieved.

By bringing $f^A C^*_{Ak} L^k$ to the left hand side of the conservation law identity (13) we get

$$f^A \left(Q^{\alpha *}_A \Delta_\alpha - C^*_{Ak} L^k \right) = D_i P^i \tag{15}$$

which still is an identity for arbitrary functions u^α, f^A. Applying the Euler operator with respect to f^A (for its definition see e.g. [3],[1]) to the left hand side of (15) gives the coefficient of f^A and on the right hand side gives zero as it is a divergence,[2] i.e. we get

$$Q^{\alpha *}_A \Delta_\alpha = C^*_{Ak} L^k \qquad \text{identically in } u^\alpha \text{ for all } A. \tag{16}$$

The vanishing of $D_i P^i$ on the right hand side of (14) was therefore not accidental.

For the Liouville equation the identity (16) takes the form

$$(u_x - D_x)\Delta \;=\; -D_y L \;=\; 0 \qquad \text{with} \tag{17}$$

$$L \;=\; u_{xx} - {u_x}^2/2. \tag{18}$$

Integrating at first (17) to $L = L(x)$ leaves the Riccati ODE

$$u_{xx} - u_x^2/2 = L(x) \tag{19}$$

for u_x to be solved, for example, through a linearizing transformation $u(x, y) = -2\log(v(x, y))$.

Output
The result of the procedure are expressions $Q^{\alpha *}_A, C^*_{Ak}$ and L^k. The relation

$$C^*_{Ak} L^k = 0 \tag{20}$$

is a necessary condition which can be solved by first regarding L^k as dependent variables and then solving

$$L^k = L^k(u^\alpha_J) \tag{21}$$

for $u^\alpha = u^\alpha(x^i)$. The system (20), (21) is a sufficient condition for the original system $\Delta_\alpha = 0$ if $Q^{\alpha *}_A$ is an invertible algebraic operator and it is a complete linearizing point transformation if (21) is purely algebraic in u^α.

5. Scope of the Procedure

The degree to which the original system $\Delta_\alpha = 0$ can be linearized depends on properties of the conservation law identity that has been computed: the number of functions f^A and the number of variables each f^A depends on, the differential order of derivatives of f^A with respect to x^i, u^α_J in C_k and in Q^α. Some properties, like the size of the solution space of remaining conditions (4) are essentially independent

[2] To prove this statement without Euler operator we could choose the f^A to be zero outside some region R such that an integral over a volume with boundary outside R will vanish using Gauss law on the right hand side of identity (15) as P^i are linear homogeneous in the f^A. Because the f^A are arbitrary inside R the coefficients of the f^A on the left hand side must vanish identically.

of the extent to which these conditions are solved. Other criteria, like the number of functions f^A and the number of their arguments does depend on the extent to which conditions (4) were solved. The strength of the procedure to be presented is to be able to handle a wide range of situations.

The following is a list of four scenarios, sorted from most special, fully algorithmic (and most beneficial) to most general, not strictly algorithmic (and less beneficial). We refer to the computational steps described in section 4 as 'the procedure'.

- If the following criteria are met:
  1. the size of the solution space of $0 = \Delta_\alpha$ is equal to the size of the solution space of $0 = C_k$,
  2. the conditions $0 = C_k$ involve q functions f^A, (equal to the number of functions u^α in Δ_α) and all f^A depend on p variables (equal to the number of variables u^α depend on),
  3. the functions Q^α expressed in terms of f^A involve f^A only algebraically, i.e. no derivatives of f^A,
  4. functions f^A do not depend on jet variables u^α_J, i.e. $f^A = f^A(x^i)$,

  then the procedure will algorithmically provide a linearization of the system $\Delta_\alpha = 0$.

  Example 2: The Burgers equation in the form

  $$0 = \Delta_1 := u_t - u_{xx} - uu_x \tag{22}$$

  for a function $u(x,t)$ can not be linearized but in the potential form

  $$0 = \Delta_2 := v_t - v_{xx} - {v_x}^2/2 \tag{23}$$

  for $v(x,t)$ a conservation law identity involving a function $f(x,t)$ is given through

  $$fe^{v/2}\Delta = D_t\left(2fe^{v/2}\right) + D_x\left(2f_xe^{v/2} - fe^{v/2}v_x\right) + 2e^{v/2}(-f_t - f_{xx}) \tag{24}$$

  and the related linearization is

  $$\begin{aligned} L &= 2e^{v/2} \\ e^{v/2}\Delta &= L_t - L_{xx} = 0. \end{aligned}$$

  A proof that every non-linear PDE (-system) that is linearizable through point or contact transformations can be linearized this way will be given in [1].
- If criteria 1,2,3 are satisfied but not 4 then a linearization is possible but at the price of a change of variables, which will be a point or contact transformation if it is invertible or otherwise it will be a non-invertible transformation depending on derivatives of u^α. Furthermore, in all such cases the transformation can be derived explicitly from the conservation law identity as will be shown in [1].
- If criterion 3 is not satisfied then the partially or completely linearized equations may only be a necessary but not a sufficient condition for $\Delta_\alpha = 0$.

- If criterion 1 is satisfied but not 2 then
  - if functions f^A of fewer than p variables occur then one can add extra variable dependencies through step 1 of the procedure,
  - if more than q functions f^A occur in $0 = C_k$ or functions f^A of more than p variables occur then one has to integrate more of the conditions $0 = C_k$ in order to be able to linearize the original system completely (a full treatment of this case will be given in [1]).
- If criterion 1 is not satisfied but the solution space of C_k involves at least one arbitrary function of one argument then the method will result in a differential expression for u^α_J which vanishes modulo $0 = \Delta_\alpha$ and factorizes into a linear differential operator acting on a non-linear differential expression. Typically this leads to a PDE for u^α which is lower in differential order than Δ_α for one of the x^i. In example 1 in section 4 and examples 8,9,10 in the appendix an equation in one less variable results, i.e. an ODE.

The algorithmic beauty of the procedure is that the above wide range of situations are covered by one and the same 4-step algorithm.

The case that a non-local linearization exists in which the L^k depend on integrals of u^α is not covered directly as the computer algebra package ConLaw does not compute non-local conservation laws. On the other hand single conservation laws (without parametric functions) can be used to introduce potentials such that the original system re-formulated in these potentials is linearizable. This approach has been successful in all 6 linearizable evolutionary systems found in [4]. Examples given in this paper are the system (25), (26) in the section 7, the system (41), (42) in section 9 and the system (57), (58) in the appendix.

6. Computational Aspects

Given a nonlinear PDE system $0 = \Delta_\alpha$, what are possible computational hurdles to be overcome in oder to find a linearization? The method described in section 4 is algorithmic and does not pose a problem. The formulation of conservation law conditions (3) and their analysis through computing a differential Gröbner basis $0 = C_k$ is algorithmic as well and could only become difficult because of a growing size of equations.

A first computational challenge lies in the fact that for linearizable systems $0 = \Delta_\alpha$ *the conservation law conditions (3) have a general solution involving arbitrary functions.* It is well known that systems of equations with a large solution space are much harder to solve than systems with only few solutions or no solutions. To incorporate many solutions, algebraic Gröbner bases for algebraic systems have to be of high degree and differential Gröbner bases for differential systems have to be of sufficiently high differential order. As a consequence, the length of expressions encountered during the Gröbner basis computation is more likely to explode and exceed available resources.

The second challenge is to integrate a Gröbner basis $0 = C_k$ *sufficiently often to meet criterion 2 in section* 5. Because the general solution of the conservation law conditions involves arbitrary functions, any integrations to be done can only be integrations of PDEs, not of ODEs.

The package CRACK that is used to compute the examples in this paper differs from similar other programs (as listed in [2]) in that it has a number of modules addressing the above problems. For example, the growth of expressions is lowered by a module for reducing the length of equations by replacing them through a suitable linear combination of equations as described in [7]. Integrations are handled by a module that integrates exact PDEs, that is able to introduce potentials to integrate certain generalizations of exact PDEs and that determines monomial integrating factors to achieve integration (see [8]). A relatively new module applies syzygies that result as a by-product of a differential Gröbner basis computation. This module allows to perform integrations more efficiently and to avoid a temporary explosion of the number of functions of integration generated in the process (see [9]). A module that integrates underdetermined linear ODEs with non-constant coefficients is often useful in the last stages of the computation. A description of the algorithm and its implementation is in preparation.

7. An Example requiring the Introduction of a Potential

The following example demonstrates that a linearization of a non-linear equation or system may only be possible if it is reformulated in terms of potentials which in turn might be found by studying conservation laws.

Example 3: The system

$$0 = \Delta_1 := u_t - u_{xx} - 2vuu_x - 2(a+u^2)v_x - v^2u^3 - bu^3 - auv^2 - cu \quad (25)$$
$$0 = \Delta_2 := v_t + v_{xx} + 2uvv_x + 2(b+v^2)u_x + u^2v^3 + av^3 + bvu^2 + cv \quad (26)$$

with $u = u(x,t),\ v = v(x,t)$ and constants a, b, c results as one of 15 cases of a class of generalized non-linear Schrödinger equations [4]. This system itself does not have conservation laws involving arbitrary functions but it has the zeroth order conservation law

$$v\Delta_1 + u\Delta_2 = D_t(uv) + D_x(v_xu - u_xv + bu^2 - av^2)$$

which motivates the introduction of a function $w(x,t)$ through

$$\begin{aligned} w_x &= uv, \\ -w_t &= v_xu - u_xv + bu^2 - av^2. \end{aligned} \quad (27)$$

The remaining system to be solved for $r := u/v$ and w simplifies if we substitute

$$w = \frac{1}{2}\log z \quad (28)$$

with $z = z(x,t)$. This substitution is not essential for the following but it reduces the size of the resulting system for $r(x,t), z(x,t)$ and eases memory requirements in the computation of conservation laws of the resulting system Δ_3, Δ_4:

$$\begin{aligned} 0 \;=\; \Delta_3 \;&:=\; 2rr_t z_x^2 + r_x^2 z_x^2 + 2ar^2 r_x z_x^2 - 2br_x z_x^2 + 2r^2 z_x z_{xxx} \\ &\quad -r^2 z_{xx}^2 + 2ar^3 z_x z_{xx} + 2brz_x z_{xx} + 4cr^2 z_x^2 \\ 0 \;=\; \Delta_4 \;&:=\; r_x z_x + rz_t - ar^2 z_x + bz_x. \end{aligned} \tag{29}$$

The program ConLaw finds a conservation law with integrating factors

$$\begin{aligned} Q^3 &= r^{-5/2} z_x{}^{-3/2} (fr + \tilde{f}) \\ Q^4 &= r^{-5/2} z_x{}^{-3/2} \left(-2z_x r(f_x r - \tilde{f}_x) - r_x z_x (fr + \tilde{f}) + z_{xx} r(fr - \tilde{f})\right) \end{aligned}$$

involving two functions $f(x,t), \tilde{f}(x,t)$ that have to satisfy the conditions

$$\begin{aligned} 0 \;=\; C_1 \;&:=\; -f_t + f_{xx} + cf - 2a\tilde{f}_x \\ 0 \;=\; C_2 \;&:=\; \tilde{f}_t + \tilde{f}_{xx} + c\tilde{f} - 2bf_x. \end{aligned}$$

The conservation law identity takes the form

$$Q^3\Delta_3 + Q^4\Delta_4 = D_t P^t + D_x P^x + L^1 C_1 + L^2 C_2 \tag{30}$$

with some conserved current (P^t, P^x) and coefficients L^1, L^2 of C_1, C_2

$$L^1 = 4\sqrt{z_x r}, \quad L^2 = 4\sqrt{z_x/r}. \tag{31}$$

Derivatives $f_x, \tilde{f}_x$ in Q^4 can be eliminated by adding total x-derivatives

$$D_x \left(r^{-5/2} z_x{}^{-3/2} 2z_x r(fr - \tilde{f})\Delta_4\right)$$

to the left hand site of the identity (30) and to $D_x P^x$. The modified form of the identity (30) is

$$\begin{aligned} 0 &= z_x^{-3/2} r^{-5/2} \left(2z_x r(fr - \tilde{f}) D_x\Delta_4 - 2r_x z_x (fr - \tilde{f})\Delta_4 + (fr + \tilde{f})\Delta_3\right) \\ &= D_t \left(4\sqrt{z_x/r}(rf - \tilde{f})\right) \\ &\quad + D_x \Big(2z_x{}^{-1/2} r^{-3/2} (-2f_x z_x r^2 - 2\tilde{f}_x z_x r + r_x z_x fr - r_x z_x \tilde{f} \\ &\qquad\qquad + z_{xx} fr^2 + z_{xx}\tilde{f}r + 4z_x a\tilde{f}r^2 + 4z_x bfr)\Big) \\ &\quad + L^1 C_1 + L^2 C_2. \end{aligned}$$

Partial integration of $L^1C_1 + L^2C_2$ until $f, \tilde{f}$ appear purely algebraically makes necessarily $P^t = P^x = 0$. Because $f, \tilde{f}$ are free we obtain the identities

$$0 = r^{-3/2} z_x{}^{-3/2} \left(\Delta_3 + 2rz_x D_x\Delta_4 - 2r_x z_x \Delta_4\right) = L_t^1 + L_{xx}^1 + cL^1 + 2bL_x^2 \tag{32}$$

$$0 = r^{-5/2} z_x{}^{-3/2} \left(\Delta_3 - 2rz_x D_x\Delta_4 + 2r_x z_x \Delta_4\right) = -L_t^2 + L_{xx}^2 + cL^2 + 2aL_x^1 \tag{33}$$

completing the linearization. For any solution L^1, L^2 of (32), (33), equations (31) provide r, z_x. With z_t from (29) we get z as a line integral, w from (28) and u, v from r and equation (27).

In the following section the effect of our method on PDEs is investigated that linearize to inhomogeneous equations.

8. Inhomogeneous Linear DEs

If the general solution of conservation law determining equations involves a number of free constants or free functions then individual conservation laws are obtained by setting all but one to zero. The remaining terms are homogeneous in the surviving constant or function. The question arises whether our conservation law based method is suitable to find linearizations that lead to linear but inhomogeneous equations.

Example 4: For the (ad hoc constructed) equation

$$0 = \Delta := 2uu_t + 2uu_{xx} + 2{u_x}^2 + 1 \tag{34}$$

the conservation law identity

$$\begin{aligned}(f_x + \tilde{f}_t)\Delta \quad = \quad & D_t\left(f_x u^2 + \tilde{f}_t u^2 + \tilde{f}\right) + \\ & D_x\left(-f_{xx}u^2 + 2f_x u u_x - \tilde{f}_{t,x}u^2 + 2\tilde{f}_t u u_x + f\right) + \\ & u^2(f_{t,x} - f_{xxx} - \tilde{f}_{txx} + \tilde{f}_{tt})\end{aligned}$$

involves functions $f(x,t), \tilde{f}(x,t)$ and establishes a conservation law provided $f, \tilde{f}$ satisfy

$$0 = f_{t,x} - f_{xxx} - \tilde{f}_{txx} + \tilde{f}_{tt}.$$

Our method gives the linear system

$$\begin{aligned} 0 &= D_x\Delta = L_{tx} + L_{xxx} && (35)\\ 0 &= D_t\Delta = L_{tt} + L_{txx}. && (36)\\ L &= u^2 \end{aligned}$$

The system (35), (36) represents the x and t derivatives of the linear equation

$$0 = L_t + L_{xx} + 1 \tag{37}$$

which is equivalent to equation (34) and is an inhomogeneous linear PDE. Although our linearization method does not quite reach (37), it nevertheless provides $L = u^2$ as the new unknown function which makes it easy to get to the equivalent linear equation (37) through a change of dependent variables in (34) or through an integration of (35), (36).

The way how homogeneous consequences can be derived from an inhomogeneous relation is to divide the inhomogeneous relation through the inhomogeneity, i.e. to make the inhomogeneity equal 1 and then to differentiate with respect to all independent variables and to obtain a set of linear homogeneous conditions in the same way as equations (35), (36) are consequences of (37). The application in the following section leads to an inhomogeneous linear PDE with non-constant inhomogeneity.

9. An Example of a Triangular Linear System

A generalization of complete linearizability of the whole PDE system in one step is the successive linearization of one equation at a time.

Example 5: Assume a triangular system of equations, like the (ad hoc constructed) system

$$0 = \Delta_1 := u_t \tag{38}$$

$$0 = \Delta_2 := vv_t - uvv_{xx} - u{v_x}^2 \tag{39}$$

with one equation (38) involving only one function, say $u = u(x,t)$, and this equation being linear or being linearizable and a second nonlinear equation being linear or linearizable in another function $v = v(x,t)$. How can the method in section 4 be used to recognize that such a system can be solved by solving successively only linear equations?

In determining all conservation laws for this system with unknown functions v, u and with integrating factors of order zero we get apart from two individual conservation laws with pairs of integrating factors $(Q^1, Q^2) = (\frac{v^2}{u^2}, -\frac{2}{u})$ and $(\frac{xv^2}{u^2}, -\frac{2x}{u})$ only one with a free function $f(u,x)$:

$$f_u\Delta_1 = D_t f$$

which indicates the linearity of Δ_1 but not the linearity of Δ_2 in v once $u(x,t)$ is known.

The proper way of applying the method of section 4 is to compute conservation laws of $0 = \Delta_2$ alone which now is regarded as an equation for $v(x,t)$ only. The function $u(x,t)$ is assumed to be parametric and given. We obtain the identity

$$2f\Delta_2 = D_x(f_x uv^2 + fu_x v^2 - 2fuvv_x) + D_t(fv^2) - v^2(f_t + uf_{xx} + 2u_x f_x + u_{xx} f).$$

which is a conservation law if f satisfies the linear condition

$$0 = f_t + uf_{xx} + 2u_x f_x + u_{xx} f. \tag{40}$$

This provides the linearization

$$\begin{aligned} 0 = 2\Delta_2 &= L_t - uL_{xx} \\ L &= v^2. \end{aligned}$$

The reason that now a linearization of Δ_2 is reached is that in the second try u is assumed to be known and therefore $u, u_{xx}, \ldots$ are not jet-variables and hence the condition (40) has solutions, otherwise not.

Examples where this triangular linearization method is successful are the systems (17), (18) in [4]. We demonstrate the method with one of these (system (17)), the other is similar.

Example 6: The system

$$0 = \Delta_1 := u_t - u_{xx} - 4uvu_x - 4u^2v_x - 3vv_x - 2u^3v^2 - uv^3 - au \tag{41}$$

$$0 = \Delta_2 := v_t + v_{xx} + 2v^2u_x + 2uvv_x + 2u^2v^3 + v^4 + av \tag{42}$$

involves functions $u(x,t), v(x,t)$ and the constant a. The single conservation law

$$0 = v\Delta_1 + u\Delta_2 = D_t(uv) + D_x(uv_x - u_x v - u^2v^2 - v^3)$$

motivates the introduction of a function $w(x,t)$ through

$$\begin{aligned} w_x &= uv & (43)\\ -w_t &= uv_x - u_x v - u^2v^2 - v^3. & (44)\end{aligned}$$

Substitution of u from (43) brings equations (42) and (44) in the form

$$\begin{aligned} 0 = \Delta_3 &:= w_t - \frac{1}{v}(-2v_x w_x + w_{xx}v + {w_x}^2 v + v^4) & (45)\\ 0 = \Delta_4 &:= v_t + v_{xx} + 2w_{xx}v + 2{w_x}^2 v + av + v^4. & (46)\end{aligned}$$

This system obeys conservation laws that involve a function $f(x,t)$ that has to satisfy $f_t = f_{xx} + af$. Our procedure provides the linearization

$$\begin{aligned} e^w(v\Delta_3 + \Delta_4) &= L^1_t + L^1_{xx} + aL^1 = 0 & (47)\\ L^1 &:= ve^w. \end{aligned}$$

The second linearized equation can be obtained by

- substituting $v = L^1/e^w$ into equations (45) and (46): to get the remaining condition
$$0 = \Delta_5 := w_t - w_{xx} - 3{w_x}^2 + 2w_x L^1_x (L^1)^{-1} - (L^1)^3 e^{-3w}, \qquad (48)$$
- assuming L^1 has been solved from (47) and treating $L^1(x,t)$ as a parametric function when computing conservation laws for equation (48) which turn out to involve two functions that have to satisfy linear PDEs,
- performing the linearization method to find that the remaining equation (48) linearizes with $L^2 = e^{3w}$ to
$$e^{3w}\Delta_5 = L^2_t - L^2_{xx} + 2L^2_x L^1_x / L^1 - 3(L^1)^3. \qquad (49)$$

Because the condition (49) is inhomogeneous for L^2 due to the term $3(L^1)^3$, actually two homogeneous linear equations are generated which are the $x-$ and $t-$derivative of (49) divided by $3(L^1)^3$ (see previous section about linear inhomogeneous equations). But as the function $L^2 = e^{3w}$ results in this process, it is no problem to find (49) from (48) directly or from an integration of these two equations. This completes the linear triangularisation of the original problem (41),(42) to the new system (47),(49).

10. Summary

The paper starts with introducing conservation law identities as a natural way to formulate infinite parameter conservation laws.

Conservation law identities are the input to a four step procedure that returns a differential consequence of the original system together with a linear differential operator that can be factored out.

Sufficient conditions on the conservation law identity which either guarantee a complete linearization or at least a partial linearization are discussed.

The possibility to find a non-local linearization arises from the application of single (finite parameter) conservation laws with the aim to introduce potentials which satisfy infinite parameter conservation laws and thus allow a linearization.

In examples it is demonstrated how the standard procedure can lead to inhomogeneous linear PDEs and how a successive linearization of one equation at a time may be possible when the whole system can not be linearized at once.

Appendix

In the appendix we list further examples of linearizations and integrations without giving details of the calculations.

The first example of the Kadomtsev-Petviasvili equation demonstrates what our method gives when a PDE has p independent variables and the conservation law involves free functions of less than $p-1$ variables. Although the result will be less useful than in the other examples, we still include it for illustration.

Example 7: The Kadomtsev-Petviasvili equation

$$0 = \Delta = u_{tx} + u_{xxxx} + 2u_{xx}u + 2{u_x}^2 - u_{yy}$$

for $u(t,x,y)$ has four conservation laws with a zeroth order integrating factor and an arbitrary function $f(t)$ as given in [6]. We comment on one of these four with an integrating factor $f_t y^3 + 6fxy$ as the situation for the others is similar. Omitting the details we only give the result of our method:

$$\begin{aligned} L^1 &= y\left(u_{txxx}y^2 + 2u_{tx}uy^2 + u_{tt}y^2 + 2u_tu_xy^2\right. \\ &\qquad \left. -6u_tx - 6u_{xxx}x + 6u_{xx} - 12u_xux + 6u^2\right) \\ L^2 &= -u_{ty}y^3 + 3u_ty^2 + 6u_yxy - 6ux \\ y(6x\Delta - y^2D_t\Delta) &= -L^1_x - L^2_y. \end{aligned}$$

The arbitrary function $f(t)$ involves only one independent variable t and the conservation law $0 = L^1_x + L^2_y$ involves two functions L^1, L^2 and has derivatives with respect to two variables x, y and is therefore not as useful as if it would be a single total derivative.

The three following equations were shown to the author first by V. Sokolov [10] who obtained their integrations earlier and independently. We add them here to demonstrate that these results can be obtained in a straight forward procedure.

Example 8: For the equation

$$0 = \Delta := u_{xy} - e^u\sqrt{{u_x}^2 - 4}, \quad u = u(x,y) \tag{50}$$

a conservation law with an arbitrary function $f(x)$ enables to factor out D_y leaving an ODE to solve

$$(u_x - D_x)\left(\frac{\Delta}{\sqrt{u_x{}^2-4}}\right) = D_y\left(\frac{-u_{xx}+u_x{}^2-4}{\sqrt{u_x{}^2-4}}\right) \quad = \quad 0. \tag{51}$$

Another conservation law with an arbitrary function $g(y)$ gives

$$\left(u_y - \frac{u_x e^u}{\sqrt{u_x{}^2-4}} - D_y\right)\Delta = D_x\left(-u_{yy}+\frac{1}{2}u_y{}^2+\frac{1}{2}e^{2u}\right) \quad = \quad 0. \tag{52}$$

Example 9: For the equation

$$0 = \Delta := u_{xy} - \left(\frac{1}{u-x}+\frac{1}{u-y}\right)u_x u_y, \quad u = u(x,y) \tag{53}$$

a conservation law with an arbitrary function $f(x)$ similarly to the above example provides

$$\frac{y-x}{(u-x)(u-y)}\Delta + D_x\left(\frac{\Delta}{u_x}\right) = D_y\left(\frac{u_{xx}}{u_x} - \frac{2(u_x-1)}{u-x} - \frac{u}{(u-x)x}\right) \quad = \quad 0. \tag{54}$$

A second conservation law is obtained from an arbitrary function $g(y)$ and is equivalent to (54) after swapping $x \leftrightarrow y$.

Example 10: For the equation

$$0 = \Delta := u_{xy} - \frac{2}{x+y}\sqrt{u_x u_y}, \quad u = u(x,y) \tag{55}$$

a conservation law with an arbitrary function $f(x)$ gives

$$\frac{1}{(x+y)}\left(\frac{1}{\sqrt{u_x}} - \frac{1}{\sqrt{u_y}}\right)\Delta + D_x\left(\frac{\Delta}{\sqrt{u_x}}\right) = D_y\left(\frac{u_{xx}}{\sqrt{u_x}} + \frac{2\sqrt{u_x}}{x+y}\right) \quad = \quad 0. \tag{56}$$

A second conservation law is obtained from an arbitrary function $g(y)$ and is equivalent to (56) after swapping $x \leftrightarrow y$.

The final example shows a linearization of a system that resulted in classifying non-linear Schrödinger type systems in [4].

Example 11: The system

$$0 = \Delta_1 \quad := \quad u_t - u_{xx} - 2vu_x - 2uv_x - 2uv^2 - u^2 - au - bv - c \tag{57}$$

$$0 = \Delta_2 \quad := \quad v_t + v_{xx} + 2vv_x + u_x \tag{58}$$

involves functions $u(x,t), v(x,t)$ and the constants a, b, c. The trivial conservation law

$$0 = \Delta_2 = D_t(v) + D_x(v_x + u + v^2)$$

motivates the introduction of a function $w(x,t)$ through

$$w_x \quad = \quad v \tag{59}$$

$$-w_t \quad = \quad v_x + u + v^2. \tag{60}$$

Substitution of u, v from (59) and (60) brings equation (57) in the form

$$
\begin{aligned}
0 \;=\; \Delta_3 \;=\; & w_{tt} + w_t^2 - w_t a - w_{xxxx} - 4w_{xxx}w_x - 3w_{xx}^2 - 6w_{xx}w_x^2 - w_{xx}a \\
& -w_x^4 - w_x^2 a + w_x b + c.
\end{aligned}
$$

This equation admits a conservation law identity

$$
\begin{aligned}
fe^w\Delta_3 \;=\; & D_t\left[(e^w)_t f - e^w f_t - e^w f a\right] + \\
& D_x\left[-(e^w)_{xxx} f + (e^w)_{xx} f_x - (e^w)_x f_{xx} + e^w f_{xxx}\right. \\
& \quad \left. -(ae^w)_x f + ae^w f_x + be^w f\right] \\
& +e^w\left[f_{tt} + af_t - f_{xxxx} - af_{xx} - bf_x + cf\right].
\end{aligned}
$$

From this follows the linearization

$$
\begin{aligned}
e^w\Delta_3 &= L_{tt} - L_t a - L_{xxxx} - L_{xx}a + L_x b + Lc \;=\; 0 \\
L &= e^w.
\end{aligned}
$$

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