Travelling waves and conservation laws for complex mKdV-type equations

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Outline

Introduction

Traveling Waves
- The First Type of Ansatz
- The Second Type of Ansatz
- Computational Remarks

Conservation Laws
- Computational Remarks

Summary
General Complex mKdV-type Equation

\[ u_t + \alpha \bar{u} u u_x + \beta u^2 \bar{u}_x + \gamma u_{xxx} = 0 \]

for complex \( u(t, x) \), \( \alpha, \beta \) and real \( \gamma > 0 \).

\( u, \alpha, \beta \) real \( \rightarrow \) ordinary mKdV.
General Complex mKdV-type Equation

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for complex \( u(t, x) \), \( \alpha, \beta \) and real \( \gamma > 0 \).

\( u, \alpha, \beta \) real \( \rightarrow \) ordinary mKdV.

\[ u(t, x) = u_1(t, x) + i u_2(t, x), \quad \alpha = \alpha_1 + i \alpha_2, \quad \beta = \beta_1 + i \beta_2 \quad \rightarrow \]

\[ u_{1t} + ((\alpha_1 + \beta_1) u_1^2 - 2 \beta_2 u_1 u_2 + (\alpha_1 - \beta_1) u_2^2) u_{1x} \]
\[ - ((\alpha_2 - \beta_2) u_1^2 - 2 \beta_1 u_1 u_2 + (\alpha_2 + \beta_2) u_2^2) u_{2x} + \gamma u_{1xxx} = 0 \]

\[ u_{2t} + ((\alpha_2 + \beta_2) u_1^2 + 2 \beta_1 u_1 u_2 + (\alpha_2 - \beta_2) u_2^2) u_{1x} \]
\[ + ((\alpha_1 - \beta_1) u_1^2 + 2 \beta_2 u_1 u_2 + (\alpha_1 + \beta_1) u_2^2) u_{2x} + \gamma u_{2xxx} = 0 \]
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physically relevant: \( \alpha, \beta \) real
General Complex mKdV-type Equation

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for complex \( u(t, x), \alpha, \beta \) and real \( \gamma > 0 \).

\( u, \alpha, \beta \) real \( \rightarrow \) ordinary mKdV.

\[
\begin{align*}
  u(t, x) &= u_1(t, x) + iu_2(t, x), \quad \alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2 \quad \rightarrow \\
  u_1t &= ((\alpha_1 + \beta_1)u_1^2 - 2\beta_2 u_1 u_2 + (\alpha_1 - \beta_1)u_2^2)u_1x \\
  &\quad - ((\alpha_2 - \beta_2)u_1^2 - 2\beta_1 u_1 u_2 + (\alpha_2 + \beta_2)u_2^2)u_2x + \gamma u_{1xxx} = 0 \\
  u_2t &= ((\alpha_2 + \beta_2)u_1^2 + 2\beta_1 u_1 u_2 + (\alpha_2 - \beta_2)u_2^2)u_1x \\
  &\quad + ((\alpha_1 - \beta_1)u_1^2 + 2\beta_2 u_1 u_2 + (\alpha_1 + \beta_1)u_2^2)u_2x + \gamma u_{2xxx} = 0 
\end{align*}
\]

physically relevant: \( \alpha, \beta \) real

integrable: \( \beta/\alpha = 0 \) or \( 1/3 \)
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- The Second Type of Ansatz
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Summary
First Type investigated

\[ u(t, x) = U(x - ct) = a + bf(x - ct) \]

\[ a = a_1 + i a_2, \ b = b_1 + i b_2 \] are complex constants, \( f(x) \) is real
First Type investigated

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either single-peaked and vanishes for large \( x \) (i.e. a solitary wave)

or no peak and approaches different constant values for large \( |x| \) (i.e. a kink)
Second Type investigated

\[ u(t, x) = \exp(i(kx + wt + \phi))f(x - ct) \]

\( k, w, \phi \) are real constants, and \( f(x) \) is real
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same general profile (i.e. solitary waves or kinks)

Both classes of complex traveling waves contain the known
sech and tanh solutions for real \(\alpha, \beta\) and \(u\).
Symmetries

Point symmetries:

- Scaling: \( x \to \lambda x, \ t \to \lambda^3 t, \ u \to \lambda^{-1}u \)
- Time translation: \( t \to t + \epsilon \)
- Space translation: \( x \to x + \epsilon \)
- Phase rotation: \( u \to \exp(i\phi)u \)
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Summary
For travelling waves using $t \rightarrow t + \epsilon, x \rightarrow x + c\epsilon$, and $\xi = x - ct, \ U = U(\xi)$

$$\rightarrow \quad -cU' + \alpha \bar{U}UU' + \beta U^2 \bar{U}' + U''' = 0$$
For travelling waves using 
\( t \to t + \epsilon, \ x \to x + c\epsilon, \) and 
\( \xi = x - ct, \ U = U(\xi) \)

\[
\Rightarrow -cU' + \alpha \bar{U} U U' + \beta U^2 \bar{U}' + U''' = 0
\]

\( \alpha, \beta \) real: real \( U \) found straightforwardly
For travelling waves using
\( t \rightarrow t + \epsilon, \ x \rightarrow x + c\epsilon, \) and
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\( \alpha, \beta \) real: real \( U \) found straightforwardly
\( \alpha, \beta \) complex: ansatz

\[ U = a + bf(\xi), \]

\( a, b \) complex constants, \( f \) real
(to study impact of complexity, not to find most general solutions)
Simplifying phase rotations:

\[ a \rightarrow \exp(i\phi)a, \quad b \rightarrow \exp(i\phi)b, \quad f \rightarrow f \]

give 2 cases:

- \( a_2 \neq 0, \quad \beta_1 = \alpha_2 = \beta_2 = 0 \);
- \( a_2 = 0, \quad \alpha_2 + \beta_2 = 0 \)

and general solution for \( f \) for each.
Solitary Waves and Kinks III

*Solitary waves:*
single peak, i.e. $f' = 0$, $f'' \neq 0$ at $\xi = \xi_0 = 0$ (w.l.o.g.)
Solitary Waves and Kinks III

Solitary waves:
single peak, i.e. \( f' = 0, f'' \neq 0 \) at \( \xi = \xi_0 = 0 \) (w.l.o.g.)
and decay, i.e. \( f(\pm \infty) = f'(\pm \infty) = f''(\pm \infty) = 0 \)
\( \rightarrow \) 6 solutions for \( f, a_1, a_2, b_1, b_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \)
Solitary Waves and Kinks III

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*Kinks:*
different asymptotic values \( f \rightarrow f_{\pm} = \text{const.} \) for \( |\xi| \rightarrow \infty \) i.e.
w.l.o.g.

\[
f(+\infty) = -f(-\infty) = f_0, \quad f'(\pm\infty) = f''(\pm\infty) = 0
\]
Solitary waves:
single peak, i.e. \( f' = 0, f'' \neq 0 \) at \( \xi = \xi_0 = 0 \) (w.l.o.g.)
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\[
f(+\infty) = -f(-\infty) = f_0, \quad f'(\pm \infty) = f''(\pm \infty) = 0
\]

inflection at \( f'' = 0 \) at \( \xi = \xi_0 = 0 \) (w.l.o.g.)
\( \rightarrow \) 2 solutions for \( f, a_1, a_2, b_1, b_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \)
4 solutions are smooth for all $\xi$
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1 solution has a blow-up singularity at $\xi = 0$
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1 solution has a blow-up singularity at $\xi = 0$
2 solutions are smooth for all $\xi$ iff $c > 0$, since otherwise there
is a blow-up singularity at some point $\xi = \xi_0$
4 solutions are smooth for all $\xi$
1 solution has a blow-up singularity at $\xi = 0$
2 solutions are smooth for all $\xi$ iff $c > 0$, since otherwise there is a blow-up singularity at some point $\xi = \xi_0$
1 solution is continuous for all $\xi$ but has a cusp at $\xi = 0$
Summary

3 Theorems about all solutions of the complex mKdV equation

- for non-singular solitary wave solutions with single peak at \( x = ct \) and decaying to a constant value for \( |x| \to \infty \) (2 sol.
  for \( \Im(\alpha) = 0, \beta = 0 \), 2 sol. for \( \Im(\alpha + \beta = 0) \)
3 Theorems about all solutions of the complex mKdV equation

- for non-singular solitary wave solutions with single peak at $x = ct$ and decaying to a constant value for $|x| \to \infty$ (2 sol. for $\Re(\alpha) = 0, \beta = 0$, 2 sol. for $\Re(\alpha + \beta = 0)$

- with a cusp singularity at $x = ct$ and a non-singular decaying tail for $|x| \to \infty$ (1 sol. with $\alpha + \beta = 0$)
3 Theorems about all solutions of the complex mKdV equation

- for non-singular solitary wave solutions with single peak at $x = ct$ and decaying to a constant value for $|x| \to \infty$ (2 sol. for $\Re(\alpha) = 0, \beta = 0$, 2 sol. for $\Re(\alpha + \beta = 0)$
- with a cusp singularity at $x = ct$ and a non-singular decaying tail for $|x| \to \infty$ (1 sol. with $\alpha + \beta = 0$)
- with non-singular kink solutions approaching different constants for $x \to \pm\infty$ (2 sol. with $\Re(\alpha + \beta) = 0$, resp. $\Re\alpha = 0, \beta = 0$)
Solitary Waves

Figure: Generalized solitary wave solution
Kinks

Figure: Generalized kink solution
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Summary
Waves with a linear Phase

Natural generalization of traveling waves:
\[ u = \exp(i(kx + wt + \phi))U(x - ct) \]

\[ \rightarrow 0 = i(w - k^3)U + i(\alpha - \beta)k\bar{U}U^2 - (c + 3k^2)U' + \alpha\bar{U}UU' + \beta U^2\bar{U}' + i3kU'' + U''' \]

Simplification: \( U = \bar{U} = f(\xi) \) (real):

\[ 0 = f''' + ((\alpha_1 + \beta_1)f^2 - c - 3k^2)f' + (\beta_2 - \alpha_2)kf^3 \]

\[ 0 = 3kf'' + (\alpha_2 + \beta_2)f^2f' + (w - k^3)f + (\alpha_1 - \beta_1)kf^3 \]

overdetermined system for \( f(\xi), k, w, \alpha_i, \beta_i \)
Waves with a linear Phase: Results I

Non-singular solitary wave solutions with non-zero linear phase:

\begin{align*}
\text{with } \sigma &= \frac{\text{Re}(\alpha - \beta)}{\text{Im}(\alpha + \beta)} = -\frac{\text{Im}(\alpha - \beta)}{\text{Re}(\alpha + \beta)} \neq 0 \text{ and } \\
\epsilon &= \text{sgn}(\sigma(x-ct)) = \pm 1, \text{ where } c > 0, \sigma^2 > 3, \text{ or } c < 0, \sigma^2 < 3,
\end{align*}
Waves with a linear Phase: Results I

Non-singular solitary wave solutions with non-zero linear phase:
1) the well-known 1-soliton solution for the integrable case
\[ \alpha = \bar{\alpha}, \beta = 0 \]
Waves with a linear Phase: Results I

Non-singular solitary wave solutions with non-zero linear phase:

1) the well-known 1-soliton solution for the integrable case $\alpha = \bar{\alpha}, \beta = 0$

2) a solitary wave solution with a cusp singularity at $x = ct$ and a non-singular decaying tail for $|x| \to \infty$: 

$$ u(t, x) = A \exp(i\phi) \exp(-\epsilon \sqrt{c \sigma_2} - 3(\sigma_2 - 1) \sigma_2 - 3ct) \exp(-\sqrt{c \sigma_2} \sigma_2 - 3|x - ct|) $$

with $\sigma = \Re(\alpha - \beta) / \Im(\alpha + \beta) = -\Im(\alpha - \beta) / \Re(\alpha + \beta)$ $\neq 0$ and $\epsilon = \text{sgn}(\sigma(x - ct)) = \pm 1$, where $c > 0$, $\sigma_2 > 3$, or $c < 0$, $\sigma_2 < 3$, and the parameters are given by $A > 0$, $0 \leq \phi < 2\pi$. 
Waves with a linear Phase: Results I

Non-singular solitary wave solutions with non-zero linear phase:

1) the well-known 1-soliton solution for the integrable case $\alpha = \bar{\alpha}, \beta = 0$

2) a solitary wave solution with a cusp singularity at $x = ct$ and a non-singular decaying tail for $|x| \to \infty$: $|\alpha| = |\beta|$ and

$$u(t, x) = A \exp(i\phi) \exp \left( -i\epsilon \sqrt{\frac{c}{\sigma^2 - 3}} \left( x - \frac{3\sigma^2 - 1}{\sigma^2 - 3} ct \right) \right)$$

$$\exp \left( -\sqrt{\frac{c \sigma^2}{\sigma^2 - 3}} |x - ct| \right)$$

with $\sigma = \Re(\alpha - \beta)/\Im(\alpha + \beta) = -\Im(\alpha - \beta)/\Re(\alpha + \beta) \neq 0$ and $\epsilon = \text{sgn}(\sigma(x - ct)) = \pm 1$, where $c > 0$, $\sigma^2 > 3$, or $c < 0$, $\sigma^2 < 3$, and the parameters are given by $A > 0$, $0 \leq \phi < 2\pi$. 
Using $\xi = x - ct$ we see that its amplitude $|u| = A \exp(-\sqrt{c\sigma^2/(\sigma^2 - 3)|\xi|})$ has a cusp at $\xi = 0$, with an arbitrary height $A > 0$, while its phase

$\arg(u) = \phi - \epsilon \sqrt{c/(\sigma^2 - 3)}\xi + \epsilon 2(\sigma^2 + 1) \sqrt{c/(\sigma^2 - 3)^3} t$ (modulo $2\pi$) exhibits a jump discontinuity at $t \neq 0$ yet is continuous at $t = 0$. Thus, this solution is a type of singular peakon.
Waves with a linear Phase: Results III

Figure: Linear phase peakon solution at $t = 0$
Waves with a linear Phase: Results IV

Figure: Linear phase peakon solution at $t = 2$
3) Non-singular kink solutions having a non-zero linear phase:
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\[
\Im \alpha = 0, \beta = 0:
\]

\[
u(t, x) = \sqrt{\frac{3(c + 3k^2)}{\alpha}} \exp(i\phi) \exp(ik(x - (3c + 8k^2)t)) \tanh\left(\sqrt{-\frac{c + 3k^2}{2(x - ct)}}\right)
\]

where \( c < -3k^2 \), \( \alpha < 0 \), and the parameters are given by \(-\infty < k < \infty\), \( 0 \leq \phi < 2\pi \).
3) Non-singular kink solutions having a non-zero linear phase: 
\[ \Im \alpha = 0, \beta = 0: \]

\[
u(t, x) = \sqrt{\frac{3(c + 3k^2)}{\alpha}} \exp(i\phi) \exp(ik(x - (3c + 8k^2)t)) \]
\[
tanh\left(\sqrt{-\frac{(c + 3k^2)}{2(x - ct)}}\right)\]

where \( c < -3k^2, \alpha < 0, \) and the parameters are given by 
\(-\infty < k < \infty, 0 \leq \phi < 2\pi.\)

This new solution generalizes the kink solution given by 
\( k = 0 \) for the integrable case \( \alpha = \bar{\alpha}, \beta = 0. \)
Waves with a linear Phase: Results VI

Figure: Linear phase kink solution
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Summary
Computational Remarks I

To be solved:

\[
0 = f''' + ((\alpha_1 + \beta_1)f^2 - c - 3k^2)f' + (\beta_2 - \alpha_2)kf^3 \\
0 = 3kf'' + (\alpha_2 + \beta_2)f^2f' + (w - k^3)f + (\alpha_1 - \beta_1)kf^3
\]
Computational Remarks I

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\[ 0 = f''' + ((\alpha_1 + \beta_1)f^2 - c - 3k^2)f' + (\beta_2 - \alpha_2)kf^3 \]
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- (+) only ODE system
Computational Remarks I

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- (+) only ODE system
- (+) overdetermined
To be solved:

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- (+) only ODE system
- (+) overdetermined
- (-) nonlinear in \( f \)
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To be solved:

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\[ 0 = 3kf'' + (\alpha_2 + \beta_2)f^2f' + (w - k^3)f + (\alpha_1 - \beta_1)kf^3 \]

- (+) only ODE system
- (+) overdetermined
- (-) nonlinear in \( f \)
- (-) 6 unknown constants \( k, w, \alpha_i, \beta_i \) increase non-linearity
Computational Remarks II

- differential elimination of leading derivatives
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- repeated, i.e. nested case distinctions due to
  - potentially vanishing multipliers
  - algebraic factorizations
  - substitutions with potentially vanishing coefficient
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- new module in CRACK splits $0 = \sum_i A_i f^i$, $A_i = \text{const}$
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- "breadth-first search" → "depth-first search"
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- CRACK: 300 interactive steps (recorded) take 4 seconds in total on a 3GHz PC
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- CRACK: 300 interactive steps (recorded) take 4 seconds in total on a 3GHz PC
- RIFFSIMP: $> 1$ day
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conserved density: \( T(t, x, u, \bar{u}, u_x, \bar{u}_x, u_{xx}, \bar{u}_{xx}) \)
Conservation Law Conditions I

conserved density: \( T(t, x, u, \bar{u}, u_x, \bar{u}_x, u_{xx}, \bar{u}_{xx}) \)

conserved flux: \( X(t, x, u, \bar{u}, u_x, \bar{u}_x, \ldots, u_{xxxx}, \bar{u}_{xxxx}) \)
Conservation Law Conditions I

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conserved flux: \( X(t, x, u, \bar{u}, u_x, \bar{u}_x, \ldots, u_{xxxx}, \bar{u}_{xxxx}) \)

conservation law: \( D_t T + D_x X = 0 \)
Conservation Law Conditions I

conserved density: \( T(t, x, u, \bar{u}, u_x, \bar{u}_x, u_{xx}, \bar{u}_{xx}) \)

conserved flux: \( X(t, x, u, \bar{u}, u_x, \bar{u}_x, \ldots, u_{xxxx}, \bar{u}_{xxxx}) \)

conservation law: \( D_t T + D_x X = 0 \)

conserved quantity: \( C = \int_{-\infty}^{\infty} T \, dx = \text{const.} \)
Conservation Law Conditions I

conserved density: $T(t, x, u, \bar{u}, u_x, \bar{u}_x, u_{xx}, \bar{u}_{xx})$

conserved flux: $X(t, x, u, \bar{u}, u_x, \bar{u}_x, ..., u_{xxxx}, \bar{u}_{xxxx})$

conservation law: $D_t T + D_x X = 0$

conserved quantity: $C = \int_{-\infty}^{\infty} T \, dx = \text{const.}$

$x$ is real $\rightarrow \Re C$ and $\Im C$ are conserved separately
$\rightarrow$ w.l.o.g. $T, X$ real
Conservation Law Conditions II

characteristic form of conservation law:

\[ D_t T + D_x(X - \Gamma) = 2\Re((u_t + \alpha \bar{u}u u_x + \beta u^2 \bar{u}_x + u_{xxx})\bar{Q}) \]

with complex multiplier \( Q = Q_1 + iQ_2 \) computed from

\[
\frac{\delta}{\delta u} \Re((u_t + \alpha \bar{u}u u_x + \beta u^2 \bar{u}_x + u_{xxx})\bar{Q}) = 0
\]

\[
\frac{\delta}{\delta \bar{u}} \Re((u_t + \alpha \bar{u}u u_x + \beta u^2 \bar{u}_x + u_{xxx})\bar{Q}) = 0
\]
Conservation Law Conditions II

characteristic form of conservation law:

\[ D_t T + D_x (X - \Gamma) = 2\Re((u_t + \alpha \bar{u} u u_x + \beta u^2 \bar{u}_x + u_{xxx})\bar{Q}) \]

with complex multiplier \( Q = Q_1 + iQ_2 \) computed from

\[
\frac{\delta}{\delta u} \Re((u_t + \alpha \bar{u} u u_x + \beta u^2 \bar{u}_x + u_{xxx})\bar{Q}) = 0
\]

\[
\frac{\delta}{\delta \bar{u}} \Re((u_t + \alpha \bar{u} u u_x + \beta u^2 \bar{u}_x + u_{xxx})\bar{Q}) = 0
\]

The density \( T \) satisfies

\[
Q = \frac{\delta T}{\delta \bar{u}}, \quad \bar{Q} = \frac{\delta T}{\delta u}
\]

and is computed through the homotopy formula

\[
T = 2\Re \left( \bar{u} \int_0^1 Q[\lambda u] \, d\lambda \right) + D_x \Upsilon
\]
Find $Q(t, x, u, \bar{u}, u_x, \bar{u}_x, u_{xx}, \bar{u}_{xx})$ from

$$D_t Q + (2\beta - \alpha)uu_x \bar{Q} + (\bar{\alpha} - 2\bar{\beta})\bar{u}u_x Q + \bar{\alpha}\bar{u}uD_x Q$$

$$+ \beta u^2 D_x \bar{Q} + D_x^3 Q = 0$$

and its complex conjugate ($u_t, \bar{u}_t, u_{tx}, \bar{u}_{tx}, \ldots$ are replaced through complex mKdV)
Conservation Law Conditions III

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and Helmholtz integrability conditions

$$D_x \Re \frac{\partial Q}{\partial u_{xx}} = \Re \frac{\partial Q}{\partial u_x}, \quad D_x \Im \frac{\partial Q}{\partial u_{xx}} = \Im \frac{\partial Q}{\partial u_x}, \quad \Im \frac{\partial Q}{\partial u_{xx}} = 0,$$

$$D_x \Re \frac{\partial Q}{\partial u_{xx}} = \Re \frac{\partial Q}{\partial u_x}, \quad D_x \Im \frac{\partial Q}{\partial u_x} = 2\Im \frac{\partial Q}{\partial u}. $$
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\]

All conditions are splitted w.r.t. \( u_{xxx}, \bar{u}_{xxx}, u_{xxxx}, \bar{u}_{xxxx}, u_{xxxxx}, \bar{u}_{xxxxx} \).
Conserved Densities

Result: 5 cases with in total 8 conservation laws

\[ \Im \alpha = \Im \beta : \quad T = u\bar{u} \]
\[ \Im \alpha = 0, \Im \beta = 0 : \quad T = -3u_x\bar{u}_x + \frac{1}{2}(\alpha + \beta)u^2\bar{u}^2 \]
\[ T = -3tu_x\bar{u}_x + \frac{1}{2}(\alpha + \beta)tu^2\bar{u}^2 - xu\bar{u} \]
\[ \alpha = 2\beta : \quad T = u + \bar{u} \]
\[ T = i(\bar{u} - u) \]
\[ \alpha = 3\beta : \quad T = \frac{1}{2}(u^2 + \bar{u}^2) \]
\[ T = i\frac{1}{2}(\bar{u}^2 - u^2) \]
\[ \Im \alpha = 0, \beta = 0 : \quad T = i(u_x\bar{u} - \bar{u}_xu) \]
Further Results on Conservation Laws

2nd order $Q_i$: 5 cases with in total 8 conservation laws
4th order $Q_i$: 2 cases with in total 3 conservation laws
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In


details are given on the interpretation of the 11 conservation laws, their multipliers, conserved densities, corresponding fluxes, integrated densities and which of the solitary wave, kink and peakon solutions admit finite conserved quantities.
Outline

Introduction

Traveling Waves
- The First Type of Ansatz
- The Second Type of Ansatz
- Computational Remarks

Conservation Laws
- Computational Remarks

Summary
Computational Remarks I

We tried $2^{\text{nd}}$ and $4^{\text{th}}$ order multipliers:

$Q_1, Q_2(t, x, u_1, u_2, u_1x, u_2x, u_{1xx}, u_{2xx})$ and

$Q_1, Q_2(t, x, u_1, u_2, u_1x, u_2x, u_{1xx}, u_{2xx}, u_{1xxx}, u_{2xxx}, u_{1xxxx}, u_{2xxxx})$
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⇒ give low priority to multi variable integrations and have a specialized module for only direct separations
2\textsuperscript{nd} order ansatz: 53 equations, 1600 computational steps,
Computational Remarks II

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$\rightarrow$ substitutions in large equations and afterwards their simplification is costly
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\textsc{Crack} is universal but applications often need extensions which benefit future applications.
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For complex mKdV

\[ u_t + \alpha \bar{u} uu_x + \beta u^2 \bar{u}_x + \gamma u_{xxx} = 0 \]

(containing the Hirota mKdV and Sasa-Satsuma mKdV) with complex parameters \( \alpha, \beta, \gamma \).
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\[ u(t, x) = U(x - ct) = a + bf(x - ct) \]
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and solutions with linear phase

\[ u(t, x) = \exp(i(kx + wt + \phi))f(x - ct) \]

as well as conservation laws.
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  solitary waves: new complex generalizations of well known sech solutions + a new complex rational solution,

- kinks: complex generalizations of the familiar mKdV tanh solution

- derived all solitary waves and kinks with linear phase which include a new type of complex peakon whose phase has a jump where the amplitude displays a cusp

- determined all 1st order conserved densities: phase invariant counterparts of the well known mKdV conserved densities for momentum, energy, Gallilean energy + a new density for the angular twist of complex kink solutions

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The End

Thank you!