Minimal General Octonion Polynomials and Octonion Identities

OCNMP - Bad Ems 2024

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Outline

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- Is a given octonion polynomial identically zero?

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- Find all central (real) polynomials up to some degree.

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- split $L_t = [L, M]$ w.r.t. $e_i, u_i, u_{ix}, \ldots$
- solve the system of bilinear algebraic conditions for the undetermined coefficients in F, L, M to obtain integrable evolution equations $u_t = F$ together with their Lax-pair L, M.

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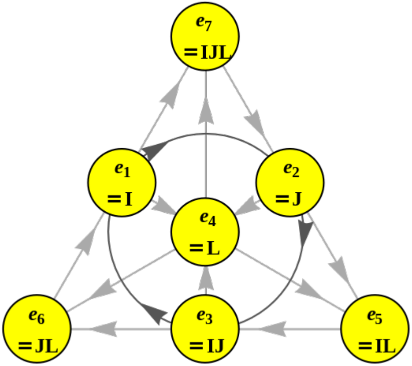
About Octonions I

- have eight dimensions
- Cayley–Dickson construction: real, complex, quaternions, octonions, sedenions,... by introducing 1 new imaginary number each time



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  as a consequence the
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  satisfies
  $[x, x, y] = [y, x, x] = 0$
  and as a consequence of that also
  $[x, y, x] = 0$.

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- Find linear combinations of identities and permutations of them that are short, highly symmetric to allow a compact formulation.

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$d = n$	1	2	3	4	5	6
m	1	1	2	5	14	42

The Computational Complexity of Repeating Factors

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i	0	0	26			

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e	1	4	28	288	3375	?

Central Polynomials

A polynomial $P = P(x, y, \ldots)$ is a *central* polynomial if P is real for any octonion variables $x, y, \ldots$ and thus commutes with any other octonian variable u:

$$[P, u] = 0$$

and thus also satisfies the vanishing identity

$$[P, u, v] = (Pv)w - P(vw) = P(vw) - P(vw) = 0$$

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Same procedure to compute them, only ignore coefficient of e_0 after splitting w.r.t. e_i.

Outline

Known Minimal Degree Central Polynomials

Racine (1986) [3], Hentzel, Peresi (1996) [4], Shestakov, Zhukavet (2009) [5]:

degree 1,2,3: None

degree 4:
$$[a, b] \circ [c, d], \tag{1}$$

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$$\sum_{\text{alt}} \{24a(b(c(de))) + 8a([b, c, d]e) - 11[a, b, [c, d, e]]\}, \tag{2}$$

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degree 6:

$$\left[\sum_{\text{alt}}\{24a(b(c(de)))+8a([b,c,d]e)-11[a,b,[c,d,e]]\},f\right]=0, \tag{5}$$

Outline

Degree 3 Vanishing Identities with Repeating Factors

Alternative laws $[u,u,v]=0, [v,u,u]=0$ give

$$[u,v,w]=[u,v,w]-[u+w,v,u+w]=...=-[w,v,u]$$

and further total antisymmetry:

$$[u,v,w]=[v,w,u]=[w,u,v]=-[v,u,w]=-[u,w,v]=-[w,v,u]$$

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This is an example for equivalence of a (not fully skey symmetric 3-variable ID to a 2-variable IDs.

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Such IDs of degree > 3 are not systematically investigated sofar but needed for reducing polynomials.

Degree 3 Minimal General Polynomials

Reductions require all identities, not only alternative laws.
$n = d = 3$ with *repeating factors*

Reductions	t	i	e
none	54	26	28
alternative laws	33	5	28
$(wu)v = ..., w > v, [wuv] = -[vuw]$	30	2	28
$(wu)v = ..., u \geq v, [wuv] = +[uvw]$	29	1	28
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List of used identities is necessary and sufficient for this purpose.

Degree 4 Minimal General Polynomials

Identities satisfied by Moufang loops (Ruth Moufang 1935) [1]

$$
\begin{aligned}
z(x(zy)) &= ((zx)z)y \\
x(z(yz)) &= ((xz)y)z \\
(zx)(yz) &= (z(xy))z \\
(zx)(yz) &= z((xy)z)
\end{aligned}
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Equivalent formulations in terms of associators:

$$w[u,v,w] = [u,vu,w] = [u,v,wu]$$
$$[u,v,w]u = [u,uv,w] = [u,v,uw]$$

Reverse Polynomials

Lemma: If P is a polynomial of octonion variables vanishing identically $P = 0$ then the reverse polynomial $R(P)$ vanishes too, $R(P) = 0$.

Example:

$$\begin{aligned} 0 &= \left(v[z,u,w] + [u,v,wz]\right)_{\{v,z\}} \\ 0 &= \left([zw,v,u] + [w,u,z]v\right)_{\{v,z\}} \end{aligned}$$

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are equivalent to

$$0 = [u,v,wz]_{\{u,w\}\{v,z\}}$$

modulo anti-symmetry of associators despite being the result of another symmetrization.

An Identity for General Non-associative Algebras

Qualitatively different:
Associator identity not using alternating property, valid for any non-associative algebra

$$0 = u[v, w, z] - [uv, w, z] + [u, vw, z] - [u, v, wz] + [u, v, w]z$$

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Not useful to remove terms but for manual proofs

Palindrome identity after $u \leftrightarrow z,\ y \leftrightarrow w$.

Degree 4 Minimal General Polynomials

Reductions require all identities, not only alternative laws.
Example: $n = d = 4$ *multilinear* case

Reductions	t	i	e
none	120	88	32
$(wu)v = ..., w > v, [wuv] = -[vuw]$	72	40	32
$(wu)v = ..., u \geq v, [wuv] = +[uvw]$	56	24	32
$(wu)v = ..., w \geq u, [wuv] = +[vwu]$	40	8	32
$(uv)(wx) = ..., v \geq x, 0 = [u, v, wz]_{\{v,w\}\{u,z\}}$	32	0	32

Reductions require all identities, not only alternative laws.
Example: $n = d = 4$ *repeating factors*

Reductions	t	i	e
none	1280	992	288
alternative laws	784	496	288
identity in 2 factor products	712	424	288
$(wu)v = ..., w > v, [wuv] = -[vuw]$	520	232	288
$(wu)v = ..., u \geq v,\ [wuv] = +[uvw]$	432	144	288
$(wu)v = ..., w \geq u, [wuv] = +[vwu]$	344	56	288
$(uv)(wx) = ..., v \geq x, 0 = [u, v, wz]_{\{v,w\}\{u,z\}}$	288	0	288

Degree 5 Minimal General Polynomials I

Reductions require all identities, not only alternative laws.
Example: $n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
none	1680	1530	150
$(wu)v = ..., w > v, [wuv] = -[vuw]$	790	640	150
$(wu)v = ..., u \geq v, [wuv] = +[uvw]$	525	375	150
$(wu)v = ..., w \geq u, [wuv] = +[vwu]$	330	180	150
$(uv)(wx) = ..., v \geq x, 0 = [u, v, wz]_{\{v,w\}\{u,z\}}$	226	76	150

Degree 5 Minimal General Polynomials II

$n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
$(pr)(u(qs)) = ..., p < q, r < s$ $0 = ([pr(u(qs))] - p(r[uqs]))_{\{pq\}\{rs\}}$	211	61	150
$0 = [p, \text{real of degree } 4]$	186	36	150
$(rp)((qs)u) = ..., p < q, r < s$ $0 = (-(rp)[qsu] + p(r[qsu]) - (ps)[rqu] + s(p[rqu]))_{\{pq\}}$	170	20	150
$(pr)(q(su)) = ..., p < q, q < r, r < s$ $0 = (+[pr(q(su))] + [pr(u(sq))]$ $-[pr(s(qu))] + p(r[qus])))_{\{pq\}\{rs\}}$	169	19	150
$(pr)((sq)u) = ..., p < q, q < r$ $0 = (-[pr((sq)u)] + [pr(q(su))] + p(u[rsq])$ $-u[(pq)rs] + u(p[qrs]))_{\{pq\}}$	167	17	150

Degree 5 Minimal General Polynomials III

$n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
$(pr)((qs)u) = ..., p < q < r < s < u$ $0 = (-[pr(s(qu))] + [pr((qs)u)] + p(u[rsq]) - u[pr(qs)])_{\{pq\}}$	166	16	150
$(qr)((ps)u), (qr)(u(ps)), (qr)(s(up)),$ based on 6 longer $(qr)(s(pu)), (qs)(u(pr)), (qu)(r(sp))$ identities	160	10	150
$p(q(r(us))), p(q(u(rs))), p(r(s(qu))), p(r(u(qs))), p(u(q(sr)))$ $p(u(r(sq))), p(s(u(qr))), q(r(p(su))), q(r(s(pu))), q(r(u(ps)))$ $0 = (q(r(u(ps))) + r(u(q(sp))) + u(q(p(sr))))_{[uq]\{qrs\}}$	150	0	150

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$(pr)((qs)u) = ..., p < q < r < s < u$ $0 = (-[pr(s(qu))] + [pr((qs)u)] + p(u[rsq]) - u[pr(qs)])_{\{pq\}}$	166	16	150
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$p(q(r(us))), p(q(u(rs))), p(r(s(qu))), p(r(u(qs))), p(u(q(sr)))$ $p(u(r(sq))), p(s(u(qr))), q(r(p(su))), q(r(s(pu))), q(r(u(ps)))$ $0 = (q(r(u(ps))) + r(u(q(sp))) + u(q(p(sr))))_{[uq]\{qrs\}}$	150	0	150

- Last reduction uses 10 identities each with 36 terms $*(*(*(**)))$:

$$0 = \Big(q(r(u(ps))) + r(u(q(sp))) + u(q(p(sr)))\Big)_{[uq]\{qrs\}}$$

$$0 = \Big(q(r(u(ps))) + r(u(s(qp))) + u(q(p(sr)))\Big)_{[ps]\{qrs\}}$$

Degree 5 Minimal General Polynomials III

$n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
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- Only left multiplications, associativity does not matter, valid for any non-associative algebra

Degree 4 Central Multilinear Polynomials

Apart from the known $[a,b]\circ[c,d]$ also this is real:

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&+p(q(rs)) + p(r(qs)) + s(r(qp)) + s(q(rp))\\
&-p(q(sr)) - p(r(sq)) - s(r(pq)) - s(q(pr))\\
&-q(p(rs)) - r(p(qs)) - r(s(qp)) - q(s(rp))\\
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Commutators of the 3 real polynomials with any octonion result in a total of 25 identities of degree 5 (table above).

Changing $p(q(rs))$ to $((pq)r)s$, $(pq)(rs)$, $(p(qr))s$, $p((qr)s)$ does not give new real polynomials.

Outline

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- Total ordering of octonion products needed to define leading terms of IDs to reduce P.
- How to avoid the extremely time-costly splitting of polynomials with, e.g. 250 million terms?
- How to lower cubic cost of solving lin. alg. system with 10^5 equations?

Algorithmic Changes

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- For multilinear P, when adding more new components to an octonian variable for the next run, set the other components of the variable temporarily to zero.
- Fine tune the number of new components per run.

Outline

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  This also provides a total ordering of all terms which is needed when using identities to remove terms from P.

General Procedure I

- Formulate most general polynomial P for given d, v. Use known octonion identities of lower degree to remove terms in the ansatz for P that could be eliminated by those identities.
- Iterate
  - When formulating P use known relations between the undetermined coefficients, initially none.
  - Expand all octonions in terms of their 8 components (initially less later increasingly more) to get in the last run τ terms.
  - Split the expanded P wrt. the 8 e_i and the (max. c) components.
  - Solve the homogeneaous linear algebraic system $P = 0$ for the unknown coefficients and substitute the solution into P.
  - Append the list of known relations between the coefficients of P.

until $P \equiv 0$ or each octonion had 8 components in the last run.

General Procedure II

- For each free coefficient c_i in the general solution of $P = 0$ print the coefficient of c_i in P which is an identically vanishing polynomial in octonion variables. There are many identities, each can have 10s to 100s of terms and may have no symmetry because any linear combination of identities is an identity.
- Pick the shortest identity, add it in a list of identities and identify the leading term (product) in this identity.
- Formulate a pattern rule to identify this leading term and similar leading terms from similar identities after permutations of varibles.
- Set the coefficients of these terms in the general ansatz for P to zero and repeat the whole procedure.
- Repeat the whole process until P is identically zero.

Discussion

- We do not stop after the first run and use all computed identities because we would not be sure whether the leading terms of these identities are independent of each other because we do not only want an (ugly, non-intelligent) list of all identities but we want a complete, necessary and sufficient set of substitution rules to be usable to simplify a general P in applications when $P = L$ or $P = M$.
- If at any stage a very short identity results that is compactifyable due to its (anti-)symmetry then start from scratch with this being the first used identity to reduce P. As a result, the identities in the final list will be shorter, more symmetric and easier to understand.

Implementation Issues II

- A first implementation as a Pascal program to generate the most general ansatz of a quaternion or octonion polynomial modulo known identities for degreee 3 and 4 is highly efficient and was used for the work reported by Philic Lam. This program does not have to expand octonions in terms of their components and is therefore not capabale to compute or verify identities.
- The first program used to do that, the Maple module 'DifferentialGeometry' evaluating a degree 4 polynomial with 1280 terms had to be stopped after 6 days.
- An implementation from scratch in the computer algebra system Reduce was able to perform such a computation in about 2 hours which is progress but still too slow for deriving degreee 5 identities with 100 times as many terms (55 million) and an even higher factor of computation time.

Implementation Issues III

- The computer algebra system Reduce has an internal total ordering of all variables. A polynomial is stored as a polynomial in the leading variables with coefficients being polynomials in the 2nd variable and so on recursively. After computing the expanded P one wants to split with respect to e_i and the with respect to components u_i. If variables c_i would have a high priority then this splitting would be an exponentially slow process for large expressions. By giving e_i highest priority and c_i lowest priority, the splitting process has only linear complexity and is extremely fast.

Efficiency Measures I

- Starting with a smaller number of components, i.e. less than $8 \times v$, and setting the other components to 0 and thus deriving just some dependencies between coefficients c_i, the total number of terms of the expanded P is even increasing because terms of P do not have a single c_i as coefficient anymore but a whole linear sum of them. But there still is a benefit. As octonion terms in P are expanded and added one by one to P, they start to cancel as the identities vanish identically.

Efficiency Measures II

- When gradually increasing the number of non-zero components of one octonion variable u then for multilinear P in the next loop the already non-zero investigated components u_j can be set to zero because due to linearity of P in u the new non-zero components do not interfere with the new non-zero components. This reduces the efford by a factor of 8. Important is a good balance between on one hand breaking up the whole computation into many small increases in the number of non-zero components in order to gather as many as possible zero c_i and relations between non-zero c_i before the last loop with the most terms and on the other hand keeping redundancy low by computing many times P that does ot provide new information on the c_j. For example, for $n = d = 5$ a good strategy is to start with 14 randomly selected non-zero components and then in the next loops to add 7,6,5,4,3,2,2,... new non-zero components until the maximum 40 is reached.

Efficiency Measures III

- The extreme version of breaking up computation would be to go through all 8^d combinations of each octonion variable having only 1 component. The number of terms at any time would then be only t and not τ. But computing ans splitting P and solving the resulting conditions 8^d times involves too much redundancy ans is too slow, although it solves any memory problems. This would be beneficial under parallel execution.

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Answer: One purpose is to avoid dropping terms that prevent the same identity. This goal is achieved if the number e of essential terms in P is unchanged after each run, i.e. after dropping the next term(s) from P.

Another purpose is to get identities in their shortest, most symmetric and thus compactifyable form.

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- new compact multilinear vanishing identities and central polynomials of degree 4, 5,
- term dropping rules applicable also to higher degree polynomials
- efficient algorithms for computations with octonions
- insight into using reverse multiplication to formulate new types of symmetries (multifactor and non-associative generalizations of the commutator and the Jordan product)

Outline

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The End

Thank you!