About Integrable Evolution Equations with Lax Pairs over the Octonions

ISQS28 — PRAGUE

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Outline

About Lax Pairs

- An evolution equation $u_t = F(u, u_x, u_{xx}, \dots)$ is said to be integrable if it has a **Lax Pair** L and M that are linear differential operators in terms of ∂_x with coefficients in $u, u_x, u_{xx}, ..$ satisfying

$$L_t = [M, L]$$

identically in all $u, u_x, u_{xx}, ..$ iff $u_t = F$.

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- Lax pair is used in the inverse scattering transform to generate multi-soliton solutions which have many physical applications.
- Two well known examples:

$u_t = uu_x + u_{xxx}$ Korteweg-De Vries (KdV) equation

$L = \partial_x^2 + \frac{1}{6}u, \qquad M = 4\partial_x^3 + u\partial_x + \frac{1}{2}u_x \qquad (w(L) = 2)$

$u_t = u^2 u_x + u_{xxx}$ modified Korteweg-De Vries (mKdV) equation

$L = \partial_x + u, \qquad M = -u_{xx} - \frac{1}{3}u^3 \qquad (w(L) = 1)$

$L = \partial_x^2 + 2u\partial_x + u^2 + u_x, \qquad M = -u_{xx} - \frac{1}{3}u^3 \qquad (w(L) = 2)$

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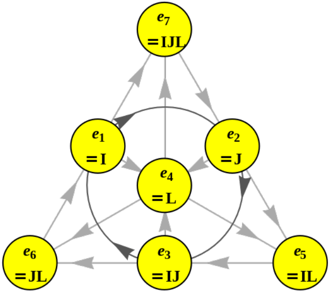
- 8-dimensional algebra formed by 1 real and 7 imaginary basis elements
- normed division algebra over the real numbers

About Octonions II

- noncommutative, nonassociative



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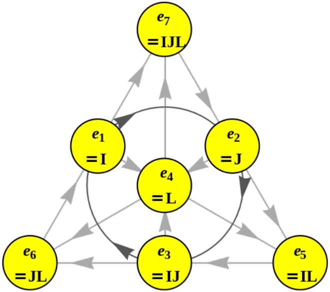

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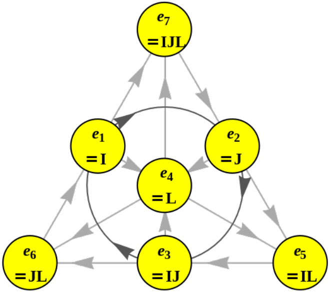

- but
  alternative: $x(xy) = (xx)y, \quad (yx)x = y(xx)$,
  as a consequence the
  associator $[x, y, z] := (xy)z - x(yz)$
  satisfies
  $[x, x, y] = [y, x, x] = 0$
  and as a consequence of that also
  $[x, y, x] = 0$.

Applications of Octonions

- appear in attempts to understand and extend the Standard Model of elementary particle physics and string theory C. Furey, Phys. Rev. D 86, 025024 (2012); T.P.Singh, Z.Naturforsch. A 75, 1051 (2020)

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- other applications in quantum logic, special relativity and supersymmetry John C. Baez https://arxiv.org/abs/math/0105155 (2002)
- little literature exists on specific integrable evolution equations over octonions A. Restuccia, A. Sotomayor, J.P. Veiro, arXiv:1609.05410v1 [math-ph] (2016)

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- Integrable scalar evolution equations have been extended to multi-component (coupled) evolution equations (e.g. over vectors, matrices, Lie algebras, graded algebras).

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- Later goal: Classification

Outline

General Idea

- Select weights for $\partial_t, \partial_x, u, L$,
  e.g. KdV-scaling: $w(\partial_t) = 3$, $w(\partial_x) = 1$, $w(u) = 2$, and start with $w(L) = 2$
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- Formulate $L_t = [M, L]$
- Split wrt. $u, u_x, u_{xx}, ...$
- Solve the overdetermined non-linear polynomial system for unknown coefficients f_j, l_j, m_j to obtain the integrable equation $u_t = F$ and Lax pair L, M.

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- To compute $(LG)_t$ replace $u_t = F, \quad u_{tx} = dF/dx, ...$
- To compute $M(LG)$ replace in MG each $G, G_x, ..$ 'in place' by $LG, (LG)_x, ...$
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$$\begin{aligned} F &= f_1 u_{xxx} + f_2 u u_x + f_3 u_x u \\ LG &= l_1 G_{xx} + l_2 G u + l_3 u G \\ MG &= m_1 G_{xxx} + m_2 G_x u + m_3 u G_x + m_4 G u_x + m_5 u_x G \end{aligned}$$

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Problem: For high $w(\partial_t), w(L)$ and low $w(\partial_x), w(u)$ the number of terms goes into the (10s of) 1000s

Adapting to Identities for Octonions

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- Find all central (real) polynomials up to some degree.

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- Find linear combinations of identities and permutations of them that are short, highly symmetric to allow a compact formulation.

The Computational Complexity of Multilinearity

$n =$ number of octonion variables $u, v, w..$ (in application $u, u_x, u_{2x}, ..$)
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τ = # of terms of P in expanded form = $t \times 8^d$
c = # of real/imag. components of all octonion variables = $8n$
i = # of identities = # of free coeff. in general solution of $P = 0$
e = # of essential terms in P which is $t - z$

$d = n$	1	2	3	4	5	6	7	8
m	1	1	2	5	14	42	132	429

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$d = n$	1	2	3	4	5	6	7	8
m	1	1	2	5	14	42	132	429
t	1	2	12	120	1680	30240	6.65×10^5	1.69×10^7

The Computational Complexity of Multilinearity

$n =$ number of octonion variables $u, v, w..$ (in application $u, u_x, u_{2x}, ..$)
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c	8	16	24	32	40	48	56	64
i	0	0	5	88	1530	?	?	?
e	1	2	7	32	150	?	?	?

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m	1	1	2	5	14	42
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i	0	0	26	992	40375	

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c	8	16	24	32	40	48
i	0	0	26	992	40375	?
e	1	4	28	288	3375	?

Central Polynomials

A polynomial $P = P(x, y, \ldots)$ is a *central* polynomial if P is real for any octonion variables $x, y, ...$ and thus commutes with any other octonian variable u:

$$[P, u] = 0$$

and thus also satisfies the vanishing identity

$$[P, u, v] = (Pv)w - P(vw) = P(vw) - P(vw) = 0$$

for any octonions u, v.

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Same procedure to compute them, only ignore coefficient of e_0 after splitting w.r.t. e_i.

Outline

Known Minimal Degree Central Polynomials

Racine (1986) [3], Hentzel, Peresi (1996) [4], Shestakov, Zhukavet (2009) [5]:

degree 1,2,3: None

degree 4:

$$[a, b] \circ [c, d], \tag{1}$$

where $x \circ y := xy + yx$,

Known Minimal Degree Central Polynomials

Racine (1986) [3], Hentzel, Peresi (1996) [4],
Shestakov, Zhukavet (2009) [5]:

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where $\sum$ is the alternating sum over the arguments.

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where $V_x(y) := x\circ y$ and $\overline{P}_3$ is defined by

$$\overline{P}_3 = V_aV_bV_c + V_cV_aV_b + V_bV_cV_a - V_bV_aV_c - V_aV_cV_b - V_cV_bV_a$$

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$$\left[\sum_{\text{alt}} \{24a(b(c(de))) + 8a([b, c, d]e) - 11[a, b, [c, d, e]]\}, f \right] = 0, \tag{5}$$

Outline

Degree 3 Vanishing Identities with Repeating Factors

Alternative laws $[u, u, v] = 0, [v, u, u] = 0$ give

$$[u, v, w] = [u, v, w] - [u + w, v, u + w] = ... = -[w, v, u]$$

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This is an example for equivalence of a (not fully skey symmetric 3-variable ID to a 2-variable IDs.

Such IDs of degree > 3 are not systematically investigated sofar but needed for reducing polynomials.

Degree 3 Minimal General Polynomials

Reductions require all identities, not only alternative laws.
$n = d = 3$ with *repeating factors*

Reductions	t	i	e
none	54	26	28
alternative laws	33	5	28
$(wu)v = ..., w > v, [w,u,v] = -[v,u,w]$	30	2	28
$(wu)v = ..., u \geq v,\ [w,u,v] = +[u,v,w]$	29	1	28
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List of used identities is necessary and sufficient for this purpose.

Degree 4 Minimal General Polynomials

Identities satisfied by Moufang loops (Ruth Moufang 1935) [1]

$$\begin{aligned} z(x(zy)) &= ((zx)z)y \\ x(z(yz)) &= ((xz)y)z \\ (zx)(yz) &= (z(xy))z \\ (zx)(yz) &= z((xy)z) \end{aligned}$$

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Equivalent formulations in terms of associators:

$$w[u,v,w] = [u,vu,w] = [u,v,wu]$$
$$[u,v,w]u = [u,uv,w] = [u,v,uw]$$

Reverse Polynomials

Lemma: If P is a polynomial of octonion variables vanishing identically $P = 0$ then the reverse polynomial $R(P)$ vanishes too, $R(P) = 0$.

Example:

$$\begin{aligned} 0 &= (v[z,u,w] + [u,v,wz])_{\{v,z\}} \\ 0 &= ([zw,v,u] + [w,u,z]v)_{\{v,z\}} \end{aligned}$$

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are equivalent to

$$0 = [u,v,wz]_{\{u,w\}\{v,z\}}$$

modulo anti-symmetry of associators despite being the result of another symmetrization.

An Identity for General Non-associative Algebras

Qualitatively different:
Associator identity not using alternating property, valid for any non-associative algebra

$$0 = u[v, w, z] - [uv, w, z] + [u, vw, z] - [u, v, wz] + [u, v, w]z$$

Not useful to remove terms but for manual proofs

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Not useful to remove terms but for manual proofs

Palindrome identity after $u \leftrightarrow z,\ y \leftrightarrow w$.

Degree 4 Minimal General Polynomials

Reductions require all identities, not only alternative laws.

Example: $n = d = 4$ *multilinear* case

Reductions	t	i	e
none	120	88	32
$(wu)v = ..., w > v, [w, u, v] = -[v, u, w]$	72	40	32
$(wu)v = ..., u \geq v,\ [w, u, v] = +[u, v, w]$	56	24	32
$(wu)v = ..., w \geq u, [w, u, v] = +[v, w, u]$	40	8	32
$(uv)(wx) = ..., v \geq x, 0 = [u, v, wz]_{\{v,w\}\{u,z\}}$	32	0	32

Degree 4 Minimal General Polynomials

Reductions require all identities, not only alternative laws.
Example: $n = d = 4$ *repeating factors*

Reductions	t	i	e
none	1280	992	288
alternative laws	784	496	288
identity in 2 factor products	712	424	288
$(wu)v = ..., w > v, [w, u, v] = -[v, u, w]$	520	232	288
$(wu)v = ..., u \geq v, \ [w, u, v] = +[u, v, w]$	432	144	288
$(wu)v = ..., w \geq u, [w, u, v] = +[v, w, u]$	344	56	288
$(uv)(wx) = ..., v \geq x, 0 = [u, v, wz]_{\{v,w\}\{u,z\}}$	288	0	288

Degree 5 Minimal General Polynomials I

Reductions require all identities, not only alternative laws.
Example: $n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
none	1680	1530	150
$(wu)v = ..., w > v, [w,u,v] = -[v,u,w]$	790	640	150
$(wu)v = ..., u \geq v, [w,u,v] = +[u,v,w]$	525	375	150
$(wu)v = ..., w \geq u, [w,u,v] = +[v,w,u]$	330	180	150
$(uv)(wx) = ..., v \geq x, 0 = [u,v,wz]_{\{v,w\}\{u,z\}}$	226	76	150

Degree 5 Minimal General Polynomials II

$n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
$(pr)(u(qs)) = ..., p < q, r < s$ $0 = ([pr(u(qs))] - p(r[uqs]))_{\{pq\}\{rs\}}$	211	61	150
$0 = [p, \text{real of degree } 4]$	186	36	150
$(rp)((qs)u) = ..., p < q, r < s$ $0 = (-(rp)[qsu] + p(r[qsu]) - (ps)[rqu] + s(p[rqu]))_{\{pq\}}$	170	20	150
$(pr)(q(su)) = ..., p < q, q < r, r < s$ $0 = (+[pr(q(su))] + [pr(u(sq))]$ $-[pr(s(qu))] + p(r[qus])))_{\{pq\}\{rs\}}$	169	19	150
$(pr)((sq)u) = ..., p < q, q < r$ $0 = (-[pr((sq)u)] + [pr(q(su))] + p(u[rsq])$ $-u[(pq)rs] + u(p[qrs]))_{\{pq\}}$	167	17	150

Degree 5 Minimal General Polynomials III

$n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
$(pr)((qs)u) = ..., p < q < r < s < u$ $0 = (-[pr(s(qu))] + [pr((qs)u)] + p(u[rsq]) - u[pr(qs)])_{\{pq\}}$	166	16	150
$(qr)((ps)u), (qr)(u(ps)), (qr)(s(up)),$ based on 6 longer $(qr)(s(pu)), (qs)(u(pr)), (qu)(r(sp))$ identities	160	10	150
$p(q(r(us))), p(q(u(rs))), p(r(s(qu))), p(r(u(qs))), p(u(q(sr)))$ $p(u(r(sq))), p(s(u(qr))), q(r(p(su))), q(r(s(pu))), q(r(u(ps)))$ $0 = (q(r(u(ps))) + r(u(q(sp))) + u(q(p(sr))))_{[uq]\{qrs\}}$	150	0	150

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- Last reduction uses 10 identities each with 36 terms $*(*(*(**)))$:

$$0 = \big(q(r(u(ps))) + r(u(q(sp))) + u(q(p(sr)))\big)_{[uq]\{qrs\}}$$

$$0 = \big(q(r(u(ps))) + r(u(s(qp))) + u(q(p(sr)))\big)_{[ps]\{qrs\}}$$

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- Only left multiplications, associativity does not matter, valid for any non-associative algebra

Degree 4 Central Multilinear Polynomials

Apart from the known $[a,b]\circ[c,d]$ also this is real:

$$\begin{aligned}
&+p(q(rs)) + p(r(qs)) + s(r(qp)) + s(q(rp))\\
&-p(q(sr)) - p(r(sq)) - s(r(pq)) - s(q(pr))\\
&-q(p(rs)) - r(p(qs)) - r(s(qp)) - q(s(rp))\\
&+q(p(sr)) + r(p(sq)) + r(s(pq)) + q(s(pr))
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= \quad & +p(q[rs]) + p(r[qs]) + s(r[qp]) + s(q[rp]) \\
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& -q(p[rs]) - r(p[qs]) - r(s[qp]) - q(s[rp]) \\
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Degree 4 Central Multilinear Polynomials continued

Similarly to $\big(p(q(rs))\big)_{[p,q],[rs],\{q,r\}\{p,s\}}$

also $\big(p(q(rs))\big)_{[p,q],[rs],\{p,r\}\{q,s\}}$

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Commutators of the 3 real polynomials with any octonion result in a total of 25 identities of degree 5 (table above).

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Commutators of the 3 real polynomials with any octonion result in a total of 25 identities of degree 5 (table above).

Changing $p(q(rs))$ to $((pq)r)s$, $(pq)(rs)$, $(p(qr))s$, $p((qr)s)$ does not give new real polynomials.

Outline

General

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- How to avoid the extremely time-costly splitting of polynomials with, e.g. 250 million terms?

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- CA systems know non-commutativity but not non-associativity.
- Why not using existing packages, like 'DifferentialGeometry' in MAPLE?
- Total ordering of octonion products needed to define leading terms of IDs to reduce P.
- How to avoid the extremely time-costly splitting of polynomials with, e.g. 250 million terms?
- How to lower cubic cost of solving lin. alg. system with 10^5 equations?

Algorithmic Changes

- For each row in the tables do not do 1 run but a sequencee of them. Start with smaller number of components than $8n$ and increase it successively.

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- Fine tune the number of new components per run.

Outline

Result for KdV-Weights

- $w(\partial_t) = 3,\ w(\partial_x) = 1,\ w(u) = 2,\ w(L) = 2$
- Two solutions have the same evolution equation

$$u_t = u_{xxx} + uu_x + u_x u = u_{xxx} + (u^2)_x$$

- Two slightly different Lax pairs

$$LG = G_{xx} + \frac{1}{3}uG, \quad MG = 4G_{xxx} + 2uG_x + u_x G$$

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- Result: 2 evolution equations each with 3 Lax pairs

Result for mKdV-Weights II

$$u_t = u_{xxx} + \alpha(u^2 u_x + u u_x u + u_x u^2) + [u, [u_x, u]]$$

$$LG = G_x - [G, u] + 2Gu$$
$$MG = [G, u_{xx}] - 2Gu_{xx} + 2[G, u_x, u] - [G, [u, u_x] + \alpha([G, u^3] - 2Gu^3)$$

$$LG = G_x + uG$$
$$MG = -u_{xx}G + 2[u_x, u, G] + [u, u_x]G - \alpha u^3 G$$

and a similar Lax pair with G on the left in all products.

Result for mKdV-Weights III

$$u_t = u_{xxx} + [u_{xx}, u] + \alpha(u^2 u_x + u u_x u + u_x u^2) + 2[u, [u_x, u]]$$

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$$MG = [G, u_{xx}] + 6[G, u_x, u] + 2[G, [u_x, u]] + \alpha[G, u^3]$$

$$LG = G_x - 2uG - Gu$$
$$MG = -[u_{xx}, G] + 3u_{xx}G + 6[u_x, u, G] - 2[[u_x, u], G] - \alpha([u^3, G] - 3u^3 G)$$

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$$u_t = u_{xxx} - 3(u^2 u_x + u_x u^2) \qquad \text{(mKdV equation)}$$

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$$u_t = u_{xxx} + 3u_x^2 \qquad \text{(Potential KdV equation)}$$

$$LG = G_{xx} + u_x G, \quad MG = 4G_{xxx} + 6u_x G_x + 3u_{xx} G$$

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- efficient algorithms for computations with octonions
- insight into using reverse multiplication to formulate new types of symmetries (multifactor and non-associative generalizations of the commutator and the Jordan product)

Conclusion II

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- The method is (mostly) algorithmic.
  - Inputs are $w(\partial_t), w(\partial_x), w(u), w(L)$.
  - The scaling homogeneous ansatz polynomials for F, LG, MG are generated by a separate program, which automatically uses octonion identities up to degree 4 to eliminate redundant terms of degree ≥ 4.
  - Currently Maple formulates the overdetermined systems.
  - Solution is done by Maple (simple cases) or 'Crack' (larger cases).
  - Start at the lowest possible $w(L)$ and later increase the weight to search for additional variants of evolution equation finally limited by complexity.

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  - Add complex conjugation $u, \bar{u}$ (e.g. NLS equation)

Outline

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The End

Thank you!