Towards a classification of evolution equations with Lax pairs over the octonions

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Outline

About Lax Pairs

- An evolution equation $u_t = F(u, u_x, u_{xx}, \dots)$ is said to be integrable if it has a **Lax Pair** L and M that are linear differential operators in terms of ∂_x with coefficients in $u, u_x, u_{xx}, ..$ satisfying

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- Two well known examples:

$u_t = uu_x + u_{xxx}$ Korteweg-De Vries (KdV) equation

$L = \partial_x^2 + \frac{1}{6}u, \qquad M = 4\partial_x^3 + u\partial_x + \frac{1}{2}u_x \qquad (w(L) = 2)$

$u_t = u^2 u_x + u_{xxx}$ modified Korteweg-De Vries (mKdV) equation

$L = \partial_x + u, \qquad M = -u_{xx} - \frac{1}{3}u^3 \qquad (w(L) = 1)$

$L = \partial_x^2 + 2u\partial_x + u^2 + u_x, \qquad M = -u_{xx} - \frac{1}{3}u^3 \qquad (w(L) = 2)$

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- normed division algebra over the real numbers

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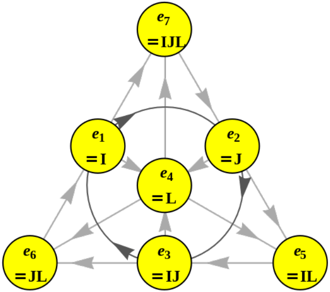
- noncommutative, nonassociative



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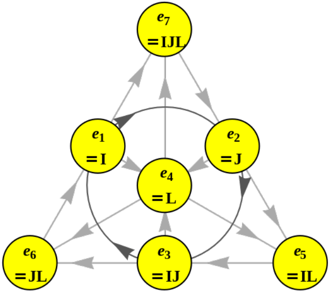

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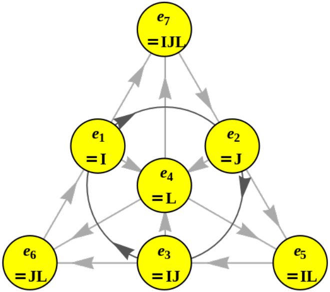

- but
  alternative: $x(xy) = (xx)y, \quad (yx)x = y(xx)$,
  as a consequence the
  associator $[x, y, z] := (xy)z - x(yz)$
  satisfies
  $[x, x, y] = [y, x, x] = 0$
  and as a consequence of that also
  $[x, y, x] = 0$.

Applications of Octonions

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- other applications in quantum logic, special relativity and supersymmetry John C. Baez `https://arxiv.org/abs/math/0105155` (2002)
- little literature exists on specific integrable evolution equations over octonions A. Restuccia, A. Sotomayor, J.P. Veiro, arXiv:1609.05410v1 [math-ph] (2016)

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- Later goal: Classification

Outline

General Idea

- Select weights for $\partial_t, \partial_x, u, L$,
  e.g. KdV-scaling: $w(\partial_t) = 3$, $w(\partial_x) = 1$, $w(u) = 2$, and start with $w(L) = 2$
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- Formulate $L_t = [M, L]$
- Split wrt. $u, u_x, u_{xx}, \ldots$
- Solve the overdetermined non-linear polynomial system for unknown coefficients f_j, l_j, m_j to obtain the integrable equation $u_t = F$ and Lax pair L, M.

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- To compute $M(LG)$ replace in MG each $G, G_x, ..$ 'in place' by $LG, (LG)_x, ...$
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Problem: For high $w(\partial_t), w(L)$ and low $w(\partial_x), w(u)$ the number of terms goes into the 100s..1000s

Adapting to Identities for Octonions

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- Find all central (real) polynomials up to some degree.

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- Find linear combinations of identities and permutations of them that are short, highly symmetric to allow a compact formulation.

The Computational Complexity of Multilinearity

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$i =$ # of identities = # of free coeff. in general solution of $P = 0$

The Computational Complexity of Multilinearity

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$e =$ # of essential terms in P which is $t - z$

$d = n$	1	2	3	4	5	6	7	8
m	1	1	2	5	14	42	132	429

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t	1	2	12	120	1680	30240	6.65×10^5	1.69×10^7

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c	8	16	24	32	40	48	56	64
i	0	0	5	88	1530	?	?	?
e	1	2	7	32	150	?	?	?

The Computational Complexity of Repeating Factors

n = number of octonion variables $u, v, w..$ (in application $u, u_x, u_{2x}, ..$)
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$d = n$	1	2	3	4	5	6
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$t =$ # of terms of $P : n^d \times m(d)$ (factors may repeat)
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$d = n$	1	2	3	4	5	6
m	1	1	2	5	14	42
t	1	4	54	1280	43750	1.95×10^6

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$d = n$	1	2	3	4	5	6
m	1	1	2	5	14	42
t	1	4	54	1280	43750	1.95×10^6
τ	8	256	9213	5.24×10^6	1.4336×10^9	5.13×10^{11}

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c	8	16	24	32	40	48

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i	0	0				

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c	8	16	24	32	40	48
i	0	0	26			

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c	8	16	24	32	40	48
i	0	0	26	992		

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t	1	4	54	1280	43750	1.95×10^6
τ	8	256	9213	5.24×10^6	1.4336×10^9	5.13×10^{11}
c	8	16	24	32	40	48
i	0	0	26	992	40375	

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c	8	16	24	32	40	48
i	0	0	26	992	40375	?
e	1	4	28	288	3375	?

Central Polynomials

A polynomial $P = P(x, y, \ldots)$ is a *central* polynomial if P is real for any octonion variables $x, y, ...$ and thus commutes with any other octonian variable u:

$$[P, u] = 0$$

and thus also satisfies the vanishing identity

$$[P, u, v] = (Pv)w - P(vw) = P(vw) - P(vw) = 0$$

for any octonions u, v.

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for any octonions u, v.

Same procedure to compute them, only ignore coefficient of e_0 after splitting w.r.t. e_i.

Outline

Known Minimal Degree Central Polynomials

Racine (1986) [3], Hentzel, Peresi (1996) [4], Shestakov, Zhukavet (2009) [5]:

degree 1,2,3: None

degree 4:
$$[a, b] \circ [c, d], \tag{1}$$

where $x \circ y := xy + yx$,

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degree 5:
$$\sum_{\text{alt}}\{24a(b(c(de))) + 8a([b,c,d]e) - 11[a,b,[c,d,e]]\}, \tag{2}$$

where $\sum$ is the alternating sum over the arguments.

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degree 6: No new ones.

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degree 1, 2: None
degree 3: Just the alternative laws
degree 4: No new ones
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$$\overline{P}_3(x^2) - \overline{P}_3(x) \circ x = 0, \tag{4}$$

where $V_x(y) := x \circ y$ and $\overline{P}_3$ is defined by

$$\overline{P}_3 = V_a V_b V_c + V_c V_a V_b + V_b V_c V_a - V_b V_a V_c - V_a V_c V_b - V_c V_b V_a$$

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degree 6:

$$\left[\sum_{\text{alt}}\{24a(b(c(de)))+8a([b,c,d]e)-11[a,b,[c,d,e]]\},f\right]=0, \tag{5}$$

Outline

Degree 3 Vanishing Identities with Repeating Factors

Alternative laws $[u, u, v] = 0, [v, u, u] = 0$ give

$$[u, v, w] = [u, v, w] - [u + w, v, u + w] = ... = -[w, v, u]$$

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$$[u, v, w] = [v, w, u] = [w, u, v] = -[v, u, w] = -[u, w, v] = -[w, v, u]$$

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This is an example for equivalence of a (not fully skey symmetric 3-variable ID to a 2-variable IDs.

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$$[u,v,w]=[u,v,w]-[u+w,v,u+w]=...=-[w,v,u]$$

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This is an example for equivalence of a (not fully skey symmetric 3-variable ID to a 2-variable IDs.

Such IDs of degree >3 are not systematically investigated sofar but needed for reducing polynomials.

Degree 3 Minimal General Polynomials

Reductions require all identities, not only alternative laws.
$n = d = 3$ with *repeating factors*

Reductions	t	i	e
none	54	26	28
alternative laws	33	5	28
$(wu)v = ..., w > v, [w,u,v] = -[v,u,w]$	30	2	28
$(wu)v = ..., u \geq v, [w,u,v] = +[u,v,w]$	29	1	28
$(wu)v = ..., w \geq u, [w,u,v] = +[v,w,u]$	28	0	28

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The 26 identities included permutations of non-(skew)symmetric identities.

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Finally $i = 0 \rightarrow$ All redundant terms from P were dropped.

List of used identities is necessary and sufficient for this purpose.

Degree 4 Minimal General Polynomials

Identities satisfied by Moufang loops (Ruth Moufang 1935) [1]

$$\begin{aligned} z(x(zy)) &= ((zx)z)y \\ x(z(yz)) &= ((xz)y)z \\ (zx)(yz) &= (z(xy))z \\ (zx)(yz) &= z((xy)z) \end{aligned}$$

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Equivalent formulations in terms of associators:

$$w[u,v,w] = [u,vu,w] = [u,v,wu]$$
$$[u,v,w]u = [u,uv,w] = [u,v,uw]$$

Reverse Polynomials

Lemma: If P is a polynomial of octonion variables vanishing identically $P = 0$ then the reverse polynomial $R(P)$ vanishes too, $R(P) = 0$.

Example:

$$\begin{aligned} 0 &= (v[z,u,w] + [u,v,wz])_{\{v,z\}} \\ 0 &= ([zw,v,u] + [w,u,z]v)_{\{v,z\}} \end{aligned}$$

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0 &= ([zw,v,u] + [w,u,z]v)_{\{v,z\}}
\end{aligned}
$$

are equivalent to

$$0 = [u,v,wz]_{\{u,w\}\{v,z\}}$$

modulo anti-symmetry of associators despite being the result of another symmetrization.

An Identity for General Non-associative Algebras

Qualitatively different:
Associator identity not using alternating property, valid for any non-associative algebra

$$0 = u[v,w,z] - [uv,w,z] + [u,vw,z] - [u,v,wz] + [u,v,w]z$$

Not useful to remove terms but for manual proofs

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Not useful to remove terms but for manual proofs

Palindrome identity after $u \leftrightarrow z,\ y \leftrightarrow w$.

Degree 4 Minimal General Polynomials

Reductions require all identities, not only alternative laws.
Example: $n = d = 4$ *multilinear* case

Reductions	t	i	e
none	120	88	32
$(wu)v = ..., w > v, [w, u, v] = -[v, u, w]$	72	40	32
$(wu)v = ..., u \geq v,\ [w, u, v] = +[u, v, w]$	56	24	32
$(wu)v = ..., w \geq u, [w, u, v] = +[v, w, u]$	40	8	32
$(uv)(wx) = ..., v \geq x, 0 = [u, v, wz]_{\{v,w\}\{u,z\}}$	32	0	32

Degree 4 Minimal General Polynomials

Reductions require all identities, not only alternative laws.
Example: $n = d = 4$ *repeating factors*

Reductions	t	i	e
none	1280	992	288
alternative laws	784	496	288
identity in 2 factor products	712	424	288
$(wu)v = ..., w > v, [w,u,v] = -[v,u,w]$	520	232	288
$(wu)v = ..., u \geq v,\ [w,u,v] = +[u,v,w]$	432	144	288
$(wu)v = ..., w \geq u, [w,u,v] = +[v,w,u]$	344	56	288
$(uv)(wx) = ..., v \geq x, 0 = [u,v,wz]_{\{v,w\}\{u,z\}}$	288	0	288

Degree 5 Minimal General Polynomials I

Reductions require all identities, not only alternative laws.
Example: $n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
none	1680	1530	150
$(wu)v = ..., w > v, [w,u,v] = -[v,u,w]$	790	640	150
$(wu)v = ..., u \geq v,\ [w,u,v] = +[u,v,w]$	525	375	150
$(wu)v = ..., w \geq u, [w,u,v] = +[v,w,u]$	330	180	150
$(uv)(wx) = ..., v \geq x, 0 = [u,v,wz]_{\{v,w\}\{u,z\}}$	226	76	150

Degree 5 Minimal General Polynomials II

$n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
$(pr)(u(qs)) = ..., p < q, r < s$ $0 = ([pr(u(qs))] - p(r[uqs]))_{\{pq\}\{rs\}}$	211	61	150
$0 = [p, \text{real of degree } 4]$	186	36	150
$(rp)((qs)u) = ..., p < q, r < s$ $0 = (-(rp)[qsu] + p(r[qsu]) - (ps)[rqu] + s(p[rqu]))_{\{pq\}}$	170	20	150
$(pr)(q(su)) = ..., p < q, q < r, r < s$ $0 = (+[pr(q(su))] + [pr(u(sq))]$ $-[pr(s(qu))] + p(r[qus])))_{\{pq\}\{rs\}}$	169	19	150
$(pr)((sq)u) = ..., p < q, q < r$ $0 = (-[pr((sq)u)] + [pr(q(su))] + p(u[rsq])$ $-u[(pq)rs] + u(p[qrs]))_{\{pq\}}$	167	17	150

Degree 5 Minimal General Polynomials III

$n = d = 5$ *multiliear* polynomial

Reductions	t	i	e
$(pr)((qs)u) = ..., p < q < r < s < u$ $0 = (-[pr(s(qu))] + [pr((qs)u)] + p(u[rsq]) - u[pr(qs)])_{\{pq\}}$	166	16	150
$(qr)((ps)u), (qr)(u(ps)), (qr)(s(up)),$ based on 6 longer $(qr)(s(pu)), (qs)(u(pr)), (qu)(r(sp))$ identities	160	10	150
$p(q(r(us))), p(q(u(rs))), p(r(s(qu))), p(r(u(qs))), p(u(q(sr)))$ $p(u(r(sq))), p(s(u(qr))), q(r(p(su))), q(r(s(pu))), q(r(u(ps)))$ $0 = (q(r(u(ps))) + r(u(q(sp))) + u(q(p(sr))))_{[uq]\{qrs\}}$	150	0	150

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- Last reduction uses 10 identities each with 36 terms $*(*(*(**)))$:

$$0 = \big(q(r(u(ps))) + r(u(q(sp))) + u(q(p(sr)))\big)_{[uq]\{qrs\}}$$

$$0 = \big(q(r(u(ps))) + r(u(s(qp))) + u(q(p(sr)))\big)_{[ps]\{qrs\}}$$

Degree 5 Minimal General Polynomials III

$n = d = 5$ *multiliear* polynomial

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- Only left multiplications, associativity does not matter, valid for any non-associative algebra

Degree 4 Central Multilinear Polynomials

Apart from the known $[a, b] \circ [c, d]$ also this is real:

$$
\begin{aligned}
&+p(q(rs)) + p(r(qs)) + s(r(qp)) + s(q(rp)) \\
&-p(q(sr)) - p(r(sq)) - s(r(pq)) - s(q(pr)) \\
&-q(p(rs)) - r(p(qs)) - r(s(qp)) - q(s(rp)) \\
&+q(p(sr)) + r(p(sq)) + r(s(pq)) + q(s(pr))
\end{aligned}
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& -q(p(rs))-r(p(qs))-r(s(qp))-q(s(rp))\\
& +q(p(sr))+r(p(sq))+r(s(pq))+q(s(pr))\\
= \quad & +p(q[rs])+p(r[qs])+s(r[qp])+s(q[rp])\\
& -q(p[rs])-r(p[qs])-r(s[qp])-q(s[rp])
\end{aligned}
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& -q(p(rs))-r(p(qs))-r(s(qp))-q(s(rp)) \\
& +q(p(sr))+r(p(sq))+r(s(pq))+q(s(pr)) \\
= \quad & +p(q[rs])+p(r[qs])+s(r[qp])+s(q[rp]) \\
& -q(p[rs])-r(p[qs])-r(s[qp])-q(s[rp]) \\
= \quad & +[p(q][rs])+[p(r][qs])+[s(r][qp])+[s(q][rp])
\end{aligned}$$

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& -p(q(sr)) - p(r(sq)) - s(r(pq)) - s(q(pr)) \\
& -q(p(rs)) - r(p(qs)) - r(s(qp)) - q(s(rp)) \\
& +q(p(sr)) + r(p(sq)) + r(s(pq)) + q(s(pr)) \\
= \; & +p(q[rs]) + p(r[qs]) + s(r[qp]) + s(q[rp]) \\
& -q(p[rs]) - r(p[qs]) - r(s[qp]) - q(s[rp]) \\
= \; & +[p(q][rs]) + [p(r][qs]) + [s(r][qp]) + [s(q][rp]) \\
= \; & \big([p(q][rs]) + [s(r][qp])\big)_{\{q,r\}}
\end{aligned}
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= \;&+[p(q][rs]) + [p(r][qs]) + [s(r][qp]) + [s(q][rp]) \\
= \;&\big([p(q][rs]) + [s(r][qp])\big)_{\{q,r\}} \\
= \;&\big((1+R)[p(q][rs])\big)_{\{q,r\}} = \big((1+R)(p(q(rs)))\big)_{[p,q],[rs],\{q,r\}}
\end{aligned}
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\end{aligned}$$

Degree 4 Central Multilinear Polynomials continued

Similarly to $\big(p(q(rs))\big)_{[p,q],[rs],\{q,r\}\{p,s\}}$

also $\big(p(q(rs))\big)_{[p,q],[rs],\{p,r\}\{q,s\}}$

is a different real degree 4 polynomial.

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is a different real degree 4 polynomial.

Commutators of the 3 real polynomials with any octonion result in a total of 25 identities of degree 5 (table above).

Degree 4 Central Multilinear Polynomials continued

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also $\big(p(q(rs))\big)_{[p,q],[rs],\{p,r\}\{q,s\}}$

is a different real degree 4 polynomial.

Commutators of the 3 real polynomials with any octonion result in a total of 25 identities of degree 5 (table above).

Changing $p(q(rs))$ to $((pq)r)s$, $(pq)(rs)$, $(p(qr))s$, $p((qr)s)$ does not give new real polynomials.

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- How to lower cubic cost of solving lin. alg. system with 10^5 equations?

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- Fine tune the number of new components per run.

Outline

Result for KdV-Weights

- $w(\partial_t) = 3,\ w(\partial_x) = 1,\ w(u) = 2,\ w(L) = 2$
- Two solutions have the same evolution equation

$$u_t = u_{xxx} + uu_x + u_x u = u_{xxx} + (u^2)_x$$

- Two slightly different Lax pairs

$$LG = G_{xx} + \tfrac{1}{3}uG, \quad MG = 4G_{xxx} + 2uG_x + u_x G$$

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- Result: 2 evolution equations each with 3 Lax pairs

Result for mKdV-Weights II

$$u_t = u_{xxx} + \alpha(u^2 u_x + u u_x u + u_x u^2) + [u, [u_x, u]]$$

$$LG = G_x - [G, u] + 2Gu$$
$$MG = [G, u_{xx}] - 2Gu_{xx} + 2[G, u_x, u] - [G, [u, u_x] + \alpha([G, u^3] - 2Gu^3)$$

$$LG = G_x + uG$$
$$MG = -u_{xx}G + 2[u_x, u, G] + [u, u_x]G - \alpha u^3 G$$

and a similar Lax pair with G on the left in all products.

Result for mKdV-Weights III

$$u_t = u_{xxx} + [u_{xx}, u] + \alpha(u^2 u_x + u u_x u + u_x u^2) + 2[u, [u_x, u]]$$

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$$MG = [G, u_{xx}] + 6[G, u_x, u] + 2[G, [u_x, u]] + \alpha[G, u^3]$$

$$LG = G_x - 2uG - Gu$$
$$MG = -[u_{xx}, G] + 3u_{xx}G + 6[u_x, u, G] - 2[[u_x, u], G] - \alpha([u^3, G] - 3u^3 G)$$

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$$u_t = u_{xxx} - 3(u^2 u_x + u_x u^2) \qquad \text{(mKdV equation)}$$

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$$u_t = u_{xxx} + 3u_x^2 \qquad \text{(Potential KdV equation)}$$

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- efficient algorithms for computations with octonions
- insight into using reverse multiplication to formulate new types of symmetries (multifactor and non-associative generalizations of the commutator and the Jordan product)

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- The method is (mostly) algorithmic.
  - Inputs are $w(u), w(\partial_x), w(\partial_t), w(L)$.
  - The scaling homogeneous ansatz polynomials for F, LG, MG are generated by a separate program, which automatically uses octonion identities up to degree 4 to eliminate redundant terms of degree ≥ 4.
  - Currently Maple formulates the overdetermined systems.
  - Solution is done by Maple (simple cases) or 'Crack' (larger cases).
  - Start at the lowest possible $w(L)$ and later increase the weight to search for additional variants of evolution equation finally limited by complexity.

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- Apply scaling freedom of L, t, x, u and $M \rightarrow M + \alpha L^n$ to reduce # of unknown coefficients

About Reverse Duality

- Reversing factors:

$$\leftrightarrow \begin{array}{c} a \cdot (b \cdot c) \\ c \cdot (b \cdot a) \end{array} \quad \text{or} \quad \leftrightarrow \begin{array}{c} (a \cdot b) \cdot (c \cdot (d \cdot e)) \\ (e \cdot d) \cdot (c \cdot (b \cdot a)) \end{array}$$

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- Symmetry broken use of identities $\rightarrow$ non-symmetric Lax pairs

$w(u)/w(\partial_x) = 2$

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			4,6	same u_t, $L_4 = {L_2}^2, L_6 = {L_2}^3$
		7	2,4	palindromic $u_t = 21$ terms, $L_4 = {L_2}^2$, hypothesis: $LG = Gu + w(\partial_t)G_{2x}, \quad LG = uG + w(\partial_t)G_{2x}$

$w(u)/w(\partial_x) = 1$

$w(u)$	$w(\partial_x)$	$w(\partial_t)$	$w(L)$	Comments
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			2	10 sol, 6 soln like for $w(L) = 1$ only with higher weight L. 4 sol for potential KdV $u_t = 3{u_x}^2 + u_{3x}$

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			3	6 sol, same u_t as for $w(L) = 1$, higher weight L

$w(u)/w(\partial_x) = 1$

$w(u)$	$w(\partial_x)$	$w(\partial_t)$	$w(L)$	Comments
1	1	3	1	6 sol, all generalized mKdV: palindromic 1. sol: $u_t = a(u_x(uu) + u(u_xu)(1+3a) + u(uu_x)) + u_{3x}$ $LG = auG + G_x$, 2. sol with reverse L, M 3. sol: $u_t = a(u_x(uu) + u(u_xu)(1+6ab^2)$ $+u(uu_x) + b[u, u_{2x}] + u_{3x}$, $LG = b[G, u] + G_x$ 4. sol like 1.,2. but slightly different L, M 5. sol like 3. but slightly different L, M 6. sol like 3. but again slightly different L, M
			2	10 sol, 6 soln like for $w(L) = 1$ only with higher weight L. 4 sol for potential KdV $u_t = 3{u_x}^2 + u_{3x}$
			3	6 sol, same u_t as for $w(L) = 1$, higher weight L
			4	14 sol, apart from all above, also solns with $u_t = -a^2(u_x(uu) + u(uu_x)) + 3u_x$ and $u_t = au_{2x}u + bu_x(uu) + cu(uu_x) + u_{3x}$ with symmetries of real limit up to order 13: $u_\tau = 3tu_{3x} + 3atuu_{2x} + 3btu^2u_x + xu_x + u$ $u_\tau = u_x$

$w(u)/w(\partial_x) = 1$ continued

$w(u)$	$w(\partial_x)$	$w(\partial_t)$	$w(L)$	Comments
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$w(u)$	$w(\partial_x)$	$w(\partial_t)$	$w(L)$	Comments
1	1	5	1	6 soln, need to be re-computed using higher degree identities to be simpler and usable
1	1	5	2	10 soln, 8 of them need to use higher degree identities in a re-computation to get simpler result, 2 soln have palindromic u_t: $u_t = u_{3x}u_x + {u_{2x}}^2 + \frac{2}{5}{u_x}^3 + u_x u_{3x} + u_{5x}$ $LG = u_x G + 5G_x$, 2. sol with reverse L, M Hypothesis: for odd $w(\partial_t)$: $LG = u_x G + w(\partial_t) G_{2x}$ + reverse L, M

$w(u)/w(\partial_x) = 1/2$

$w(u)$	$w(\partial_x)$	$w(\partial_x)$	$w(L)$	Comments
1	2	3	1	3 sol all like $u_t = au_x u + buu_x$. The special case of anti-palindromiC $u_t = [u, u_x]$ has palindromic $LG = uG + Gu$ and anti-palin. $MG = [G, u_x]$

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			2..5	3 sol, all like for $w(L) = 1$ with same symmetry properties, only with L having one more u as factor for each increased $w(L)$. Example: $L_2 = u^2G$ or $= Gu^2$ or $= a(u^2G + Gu^2) + buGu$
		4	1..5	no solutions

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		4	1..5	no solutions
		5	1	anti-palindromic: $u_t = [u, u_{2x}]$ with arbitrary multiples of products $u_x u^3$ in all multiplication orders of u, u_x being compatible
		6	1..4	(Ibragimov-Shabat) no solutions

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		7	1	$u_t = [u_{3x}, u], LG = uG, MG = u_{3x}G + 2Gu_{3x}$ and 5 other solution
		7	2	6 sol, needs re-run using higher deree identities 1 sol: $u_t = [u_{3x}, u], LG = uG, MG = u_{3x}G + 2Gu_{3x}$

$w(u)/w(\partial_x) = 3/2$

$w(u)$	$w(\partial_x)$	$w(\partial_x)$	$w(L)$	Comments
3	2	5	3	3 sol, 1: $u_t = au_xu + buu_x$, 2: rev sym soln 3: special case: anti-palindromic $u_t = [u, u_x]$ palin. $LG = uG + Gu$, anti-pal. $MG = [G, u_x]$

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			6,9,12	as above with 1 extra u in LG

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			6,9,12	as above with 1 extra u in LG
			11	1 sol, same as 3. sol of $w(L) = 3$ but with higher degree LG

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			6,9,12	as above with 1 extra u in LG
			11	1 sol, same as 3. sol of $w(L) = 3$ but with higher degree LG
		7	3,6,9,12	3 sol, all $u_t = [u, u_{2x}]$, same MG 2 sol reverse sym, the 3rd palindromic

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		7	3,6,9,12	3 sol, all $u_t = [u, u_{2x}]$, same MG 2 sol reverse sym, the 3rd palindromic
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			11,13,14	1 sol u_t like 3. sol of $w(L) = 3$

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			6,9,12	as above with 1 extra u in LG
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			6,9,12	as above with 1 extra u in LG
			11	1 sol, same as 3. sol of $w(L) = 3$ but with higher degree LG
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		11	3,6,9	3 sol with $u_t = [u, u_{4x}] + au_xu^3 + buu_xu^2$ $+cu^2u_xu + du^3u_x$

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			6,9,12	as above with 1 extra u in LG
			11	1 sol, same as 3. sol of $w(L) = 3$ but with higher degree LG
		7	3,6,9,12	3 sol, all $u_t = [u, u_{2x}]$, same MG 2 sol reverse sym, the 3rd palindromic
				2 sol with $u_t = {u_x}^2$ both reverse sym.
			11,13,14	1 sol u_t like 3. sol of $w(L) = 3$
		9	3,6,9	3 sol like $w(\partial_t) = 5, w(L) = 3$ only with $u_t = [u, u_{3x}]$
			5,10	3 sol like $w(\partial_t) = 5, w(L) = 3$ only with $u_t = [u_x, u_{2x}]$
		11	3,6,9	3 sol with $u_t = [u, u_{4x}] + au_xu^3 + buu_xu^2$ $+cu^2u_xu + du^3u_x$
		11	11	1 sol with same u_t

Conclusion IV on Observations

- Increasing $w(L)$ can show new integrable equations with same other weights (eg. mKdV $\rightarrow$ potential KdV).
- If a set of weights of $u, \partial_x, \partial_t, L$ includes an integrable equation then increasing $w(L)$ in fixed intervals gives more lax pairs for same equation.
- Increasing $w(\partial_t)$ in fixed intervals show higher symmetries of integrable equations.

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Outstanding Mathematical Challenges

- Use Lax pairs to compute other features of integrability.
- Are multiple Lax pairs useful for anything?
- Is there some upper bound for the weight (degree) of L for a Lax Pair to exit for a given integrable octonion equation?

Outline

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The End

Thank you!