SYMMETRIES AND CASIMIRS OF RADIAL COMPRESSIBLE FLUID FLOW AND GAS DYNAMICS IN $n > 1$ DIMENSIONS

STEPHEN C. ANCO AND SARA SEIFI AND THOMAS WOLF
DEPARTMENT OF MATHEMATICS, BROCK UNIVERSITY, ST. CATHARINES, ON CANADA

Abstract. Symmetries and Casimirs are studied for the Hamiltonian equations of radial compressible fluid flow in $n > 1$ dimensions. An explicit determination of all Lie point symmetries is carried out, from which a complete classification of all maximal Lie symmetry algebras is obtained. The classification includes all Lie point symmetries that exist only for special equations of state. For a general equation of state, the hierarchy of advected conserved integrals found in recent work is proved to consist of Hamiltonian Casimirs. A second hierarchy that holds only for an entropic equation of state is explicitly shown to comprise non-Casimirs which yield a corresponding hierarchy of generalized symmetries through the Hamiltonian structure of the equations of radial fluid flow. The first-order symmetries are shown to generate a non-abelian Lie algebra. Two new kinematic conserved integrals found in recent work are likewise shown to yield additional first-order generalized symmetries holding for a barotropic equation of state and an entropic equation of state. These symmetries produce an explicit transformation group acting on solutions of the fluid equations. Since these equations are well known to be equivalent to the equations of gas dynamics, all of the results obtained for n-dimensional radial fluid flow carry over to radial gas dynamics.

1. Introduction

In recent work [1] studying radial compressible fluid flow in $n > 1$ dimensions, new conserved integrals and new advected scalars (invariants) have been found which are not inherited by radial reduction from the known conserved integrals and invariants of n-dimensional non-radial fluid flow [2, 3]. These "hidden" quantities indicate that, compared to the full Euler equations governing compressible fluid flow, the radial Euler equations have a much richer structure. The same results hold for the equations of gas dynamics, which have a well-known equivalence [25] to the Euler equations of compressible fluid flow.

Two of the new conserved integrals describe kinematic quantities, one being an enthalpy-flux which holds for barotropic equations of state, and the other being an entropy-weighted energy which holds for entropic equations of state. Most interestingly, the new advected scalars comprise two infinite hierarchies that hold respectively for general non-barotropic equations of state and entropic equations of state. Both hierarchies are generated by a recursion operator applied to basic advected scalars. Each of the hierarchies gives rise to corresponding advected integrals on transported radial domains, which are obtained in terms of conserved densities derived from the advected scalars.

These unexpected results motivate the present study of the Hamiltonian structure, Casimirs, and symmetries of the equations for radial fluid flow and radial gas dynamics in $n > 1$ dimensions, with a general equation of state.

Firstly, all point symmetries will be determined, including any that exist only for special equations of state. A rich structure of symmetry algebras and attendant equations of state is seen to exist, which is far wider than the structure of kinematic conserved integrals found in Ref. [1]. The corresponding symmetry transformation groups will be described.

Secondly, all Hamiltonian Casimirs up to first-order will be determined. It is found that these coincide with the new advected integrals of zeroth and first order in the hierarchy holding for a general equation of state. An inductive proof that all of the higher-order advected integrals are Casimirs is given.

Thirdly, the remaining new advected integrals are shown to not be Casimirs. They instead give rise to generalized (non-point) symmetries through the general well-known correspondence between non-Casimir conserved integrals and Hamiltonian symmetries. The resulting symmetries of the radial compressible fluid equations include subalgebras of first-order symmetries and higher-order symmetries. This is a "hidden" symmetry structure which does not arise from radial reduction of the Hamiltonian symmetries admitted by the n-dimensional Euler equations for compressible fluid flow. The first-order symmetries are shown to produce an explicit transformation group acting on solutions of the radial equations.

The main results presented here can have physical applications to explosive flows and implosive flows (see e.g. [22, 15, 24, 14, 25, 10]), as well as numerous engineering applications (see e.g. [8, 11]). Moreover, they may be indicative of hidden structure for more general zero-vorticity flows.

The rest of this paper is organized as follows. In Section 2, the equations of radial fluid flow and their equivalence to the equations of radial gas dynamics are summarized. The Hamiltonian structure of these equations and attendant properties are presented. In Section 3, the classification of point symmetries and algebras is stated. The results on Hamiltonian Casimirs are presented in Section 4. In Section 5, the Hamiltonian symmetries are derived. Finally, some concluding remarks are made in Section 6.

Appendix A provides remarks on computational aspects of the main classifications. Appendix B summarizes some variational identities which are used in the proofs.

General results on symmetries can be found in Ref. [19, 9, 5]. See Ref. [6, 12, 21, 23, 16, 18, 17, 7] for key work on the Hamiltonian structure and Casimirs for the non-radial Euler equations.

2. Radial flow equations and Hamiltonian structure

The radial reduction of the Euler equations of compressible fluid flow in n dimensions without boundaries is given by the system

$$U_t + UU_r + (p_S S_r + p_\rho \rho_r)/\rho = 0, \tag{2.1}$$

$$\rho_t + (U\rho)_r + \tfrac{n-1}{r} U\rho = 0, \tag{2.2}$$

$$S_t + US_r = 0, \tag{2.3}$$

where $U = U(r,t)$ is the radial component of the fluid velocity, $\rho = \rho(r,t)$ is the fluid density, and $S = S(r,t)$ is the local entropy.

This system is closed by specifying an equation of state, which in general is given by

$$p = p(\rho, S). \tag{2.4}$$

The equation of state determines all thermodynamic quantities in terms of the internal energy $e(\rho, S)$ (per unit mass) through the thermodynamic relation $T\,dS = de + p\,d(1/\rho)$, where T is the local temperature (per unit mass). In particular, the internal energy is given by

$$e(\rho, S) = \int (p(\rho, S)/\rho^2)\, d\rho, \tag{2.5}$$

which determines

$$T(\rho, S) = \frac{\partial e}{\partial S}\Big|_\rho = \int (p_S(\rho, S)/\rho^2)\, d\rho. \tag{2.6}$$

The equations governing radial gas dynamics consist of the density equation (2.2) and the velocity equation formulated as

$$U_t + UU_r + p_r/\rho = 0, \tag{2.7}$$

together with the pressure equation

$$p_t + Up_r + a^2\rho(U_r + \tfrac{n-1}{r}U) = 0, \tag{2.8}$$

where

$$a = a(\rho, p) > 0 \tag{2.9}$$

is the speed of sound. The pressure equation can be derived from the equation of state (2.4) by use of the implicit function theorem to obtain $S = F(\rho, p)$, which is then substituted into the entropy equation (2.3) and simplified using the density equation (2.2), with

$$a^2 = -F_\rho/F_p = \frac{\partial p}{\partial \rho}\Big|_{S=F(\rho,p)}. \tag{2.10}$$

Conversely, the entropy equation can be recovered from the pressure equation (2.8) by solving $F_\rho + a^2(\rho, p)F_p = 0$ to obtain $S = F(\rho, p)$, which is then observed to satisfy the entropy equation via the density and pressure equations.

As a consequence, the equations of state familiar in fluid flow — barotropic, polytropic, ideal gas — and in gas dynamics — ideal gas law — have a direct correspondence in the equivalent formulations (2.1)–(2.4) and (2.2),(2.7)–(2.9). Explicit equivalences are given in Ref. [1].

2.1. **Radial Hamiltonian formulation.** The radial Euler equations (2.1)–(2.3) possess a Hamiltonian formulation which arises directly from reduction of the well-known Hamiltonian formulation [23, 3] of the n-dimensional Euler equations. It is given by

$$\partial_t \begin{pmatrix} U \\ \rho \\ S \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta H/\delta U \\ \delta H/\delta \rho \\ \delta H/\delta S \end{pmatrix} \tag{2.11}$$

with the Hamiltonian

$$H = \int_0^\infty \rho(\tfrac{1}{2}U^2 + e) r^{n-1}\, dr \tag{2.12}$$

where

$$\mathcal{H} = \begin{pmatrix} 0 & -D_r r^{1-n} & r^{1-n}\frac{1}{\rho}S_r \\ -r^{1-n}D_r & 0 & 0 \\ -r^{1-n}\frac{1}{\rho}S_r & 0 & 0 \end{pmatrix} \tag{2.13}$$

is the (non-canonical) Hamiltonian operator. A general discussion of the properties of Hamiltonian operators can be found in Ref. [19].

The Hamiltonian (2.12) physically describes a conserved energy for radial fluid flow.

The Hamiltonian structure (2.11)–(2.13) corresponds to a Poisson bracket. Specifically, for any two functionals F and G on a transported radial domain $V(t)$, their Poisson bracket is defined in terms of Hamiltonian operator (2.13) by

$$\{F, G\} = \int_{V(t)} (\nabla^{\rm t}_{U,\rho,S} F \mathcal{H} \nabla_{U,\rho,S} G) r^{n-1}\, dr \tag{2.14}$$

modulo a trivial functional, where $\nabla_{U,\rho,S} = \left(\frac{\delta}{\delta U}, \frac{\delta}{\delta \rho}, \frac{\delta}{\delta S}\right)$ denotes the variational gradient, and the "t" denotes the transpose. Note that a functional is trivial if its density is a total r-derivative whereby the functional reduces identically to a surface integral on the transported boundary $\partial V(t)$.

The Poisson bracket (2.14) is a bilinear map on the space of moving conserved integrals and has the properties that it is skew and obeys the Jacobi identity.

If a non-trivial functional F has no explicit dependence on t, then its time evolution on a fixed radial domain V is given by

$$\frac{dF}{dt} = \{F, H\} \tag{2.15}$$

modulo a trivial functional. Hence, such a non-trivial functional F will be a conserved integral if and only if $\{F, H\} = 0$.

Radial gas dynamics has a similar Hamiltonian formulation, which can be derived through the change of variables $(\rho, S) \to (\rho, p)$ together with $p(\rho, S) \to a^2(\rho, p)$ using the equation of state relation (2.10). In particular, the variational derivatives transform as

$$\frac{\delta}{\delta U}\Big|_{(U,\rho,S)} = \frac{\delta}{\delta U}\Big|_{(U,\rho,p)}, \tag{2.16a}$$

$$\frac{\delta}{\delta \rho}\Big|_{(U,\rho,S)} = \frac{\delta}{\delta \rho}\Big|_{(U,\rho,p)} + a^2 \frac{\delta}{\delta p}\Big|_{(U,\rho,p)}, \tag{2.16b}$$

$$\frac{\delta}{\delta S}\Big|_{(U,\rho,S)} = (\partial p/\partial S)|_\rho \frac{\delta}{\delta p}\Big|_{(U,\rho,p)}. \tag{2.16c}$$

It is then straightforward to see that the Hamiltonian structure (2.11) and (2.13) becomes

$$\partial_t \begin{pmatrix} U \\ \rho \\ p \end{pmatrix} = \mathcal{H}_{\rm gas} \begin{pmatrix} \delta H/\delta U \\ \delta H/\delta \rho \\ \delta H/\delta p \end{pmatrix} \tag{2.17}$$

with

$$\mathcal{H}_{\rm gas} = \begin{pmatrix} 0 & -D_r r^{1-n} & r^{1-n}\frac{1}{\rho}p_r - \frac{1}{\rho}D_r r^{1-n}\rho a^2 \\ -r^{1-n}D_r & 0 & 0 \\ -r^{1-n}\frac{1}{\rho}p_r - r^{1-n}\rho a^2 D_r \frac{1}{\rho} & 0 & 0 \end{pmatrix} \tag{2.18}$$

where the same Hamiltonian (2.12) is used, with the internal energy e expressed in terms of ρ and p. This expression can be obtained from $\frac{\partial e(\rho,S)}{\partial \rho} = \frac{\partial e(\rho,p)}{\partial \rho} + a^2 \frac{\partial e(\rho,p)}{\partial p} = p^2/\rho$, as shown by combining equation (2.5) and relation (2.16b).

The associated Poisson bracket for radial gas dynamics is likewise obtained by putting $\mathcal{H} \to \mathcal{H}_{\rm gas}$ and $\nabla_{U,\rho,S} \to \nabla_{U,\rho,p}$ in the definition (2.14).

2.2. **Conserved integrals.** The results in Ref. [1] show that the kinematic conserved integrals for the radial Euler equations (2.1)–(2.3) consist of:

$$\text{mass}\quad \frac{d}{dt}\int_{V(t)} \rho\, r^{n-1} dr = 0, \tag{2.19}$$

$$\text{total generalized-entropy}\quad \frac{d}{dt}\int_{V(t)} \rho f(S)\, r^{n-1} dr = 0, \tag{2.20}$$

$$\text{energy}\quad \frac{d}{dt}\int_{V(t)} \rho(\tfrac{1}{2}U^2 + e)\, r^{n-1} dr = -(r^{n-1}pU)\Big|_{\partial V(t)}, \tag{2.21}$$

in the case of a general equation of state $p = p(\rho, S)$, where $e = \int p(\rho, S)/\rho^2\, d\rho$;

$$\text{dilational energy}\quad \frac{d}{dt}\int_{V(t)} \big(t\rho(\tfrac{1}{2}U^2 + e) - \tfrac{1}{2}r\rho U\big)\, r^{n-1} dr = -\big(r^{n-1}(tU - \tfrac{1}{2}r)p\big)\Big|_{\partial V(t)}, \tag{2.22}$$

$$\text{similarity energy}\quad \frac{d}{dt}\int_{V(t)} \big(t^2\rho(\tfrac{1}{2}U^2 + e) - t\, r\rho U + \tfrac{1}{2}r^2\rho\big)\, r^{n-1} dr = -\big(r^{n-1}t(tU - r)p\big)\Big|_{\partial V(t)}, \tag{2.23}$$

in the case of a polytropic equation of state $p = \kappa(S)\rho^{1+2/n}$, where $e = \frac{n}{2}\kappa(S)\rho^{2/n}$ and κ is an arbitrary function;

$$\text{enthalpy flux}\quad \frac{d}{dt}\int_{V(t)} U\, dr\Big|_{\mathcal{E}} = -(e + p/\rho - \tfrac{1}{2}U^2)\Big|_{\partial V(t)} \tag{2.24}$$

for a barotropic equation of state $p = p(\rho)$, where $e = \int p(\rho)/\rho^2\, d\rho$;

$$\text{entropy-weighted energy}\quad \frac{d}{dt}\int_{V(t)} \big(\tfrac{1}{2}\rho U^2 f(S) - K(S)\big)\, r^{n-1} dr\Big|_{\mathcal{E}} = -\big(\tfrac{1}{2}r^{n-1}UK(S)\big)\Big|_{\partial V(t)} \tag{2.25}$$

for an entropic equation of state $p = \kappa(S)$, where $e = -\kappa(S)/\rho$ and $K(S) = \int f(S)\kappa'(S)\, dS$. Note f denotes an arbitrary non-constant function of its argument.

In these integrals, $V(t)$ is any radial domain that is transported in the flow. The mass (2.19) and total entropy (2.20) are advected integrals, since their net flux through the moving boundary $\partial V(t)$ is zero. The radial Euler equations (2.1)–(2.3) have been shown in Ref. [1] to possess two hierarchies of non-kinematic advected integrals.

One hierarchy holds for a general equation of state $p = p(\rho, S)$:

$$\mathcal{I}_l = \int_{V(t)} \rho f(J_0, J_1, \ldots, J_l)\, r^{n-1} dr, \quad l = 1, 2, \ldots \tag{2.26}$$

where

$$J_l = \mathcal{R}^l S, \quad l = 0, 1, 2, \ldots \tag{2.27}$$

are advected scalars given in terms of the recursion operator

$$\mathcal{R} = (r^{1-n}/\rho) D_r. \tag{2.28}$$

The conserved integral (2.26) is non-trivial at order $l \geq 1$ if and only if f is nonlinear in its last argument, namely $f_{J_l J_l} \not\equiv 0$.

Note that $\mathcal{I}_0$ is the entropy integral (2.20), since $J_0 = S$.

The other hierarchy holds only for an entropic equation of state $p = \kappa(S)$:

$$\mathcal{I}'_l = \int_{V(t)} \rho f(J_0, J_1, J_{1,1}, J_{2,1}, \ldots, J_l, J_{1,l}, J_{2,l})\, r^{n-1} dr \tag{2.29}$$

where

$$J_{1,l} = \mathcal{R}^{l-1}\big(U^2 + \tfrac{2}{n} r p_r/\rho\big), \quad l = 1, 2, \ldots \tag{2.30}$$

$$J_{2,l} = \mathcal{R}^{l-1}\big(A(r, U, p_r/\rho) - t\big) = \int_0^1 \mathcal{R}^l \Big(r/\sqrt{U^2 + \tfrac{2}{n}(1 - y^n) r p_r/\rho}\Big) dy, \quad l = 1, 2, \ldots \tag{2.31}$$

are advected scalars, with

$$A(r, U, p_r/\rho) = \int_0^r \frac{dy}{\sqrt{U^2 + \frac{2}{n}(1 - (y/r)^n) r p_r/\rho}} = \int_0^1 \frac{r\, dy}{\sqrt{U^2 + \frac{2}{n}(1 - y^n) r p_r/\rho}}. \tag{2.32}$$

Here the conserved integral (2.29) is non-trivial at order $l \geq 2$ if and only if f is nonlinear in at least one of $J_{1,l}$ and $J_{2,l}$; at order $l = 1$, it is non-trivial if and only if f is non-constant in at least one of $J_{1,1}$ and $J_{2,1}$.

Note that, as shown in Ref. [1], $\mathcal{I}'_1$ with $f = \frac{1}{2} J_{1,1}$ is equivalent (modulo a trivial integral) to the energy integral (2.21) with $p = \kappa(S) = -\rho e$.

3. Point symmetries

A Lie point symmetry of the radial Euler equations (2.1)–(2.3) is a one-parameter transformation group on (t, r, ρ, S, U) with a generator

$$\mathrm{X} = \tau \partial_t + \xi \partial_r + \eta^\rho \partial_\rho + \eta^S \partial_S + \eta^U \partial_U \tag{3.1}$$

whose coefficients are functions of (t, r, ρ, S, U) such that the solution space $\mathcal{E}$ of the equations is mapped into itself. The transformation group can be obtained from the generator via $(t, r, \rho, S, U) \to \exp(\epsilon \mathrm{X})(t, r, \rho, S, U)$ where ϵ is the group parameter.

The action of a symmetry transformation group on solutions $(U(t, r), \rho(t, r), S(t, r))$ is given by

$$\begin{aligned} U(t, r) &\to U(t, r) + \epsilon P^U|_{(U(t,r), \rho(t,r), S(t,r))} + O(\epsilon^2), \\ \rho(t, r) &\to \rho(t, r) + \epsilon P^\rho|_{(U(t,r), \rho(t,r), S(t,r))} + O(\epsilon^2), \\ S(t, r) &\to S(t, r) + \epsilon P^S|_{(U(t,r), \rho(t,r), S(t,r))} + O(\epsilon^2), \end{aligned} \tag{3.2}$$

where

$$P^\rho = \eta^\rho - \tau \rho_t - \xi \rho_r, \quad P^U = \eta^U - \tau U_t - \xi U_r, \quad P^S = \eta^S - \tau S_t - \xi S_r. \tag{3.3}$$

The infinitesimal form of this action is given by a generator

$$\hat{\mathrm{X}} = P^\rho \partial_\rho + P^S \partial_S + P^U \partial_U \tag{3.4}$$

which is called the characteristic form of the symmetry.

The solution space $\mathcal{E}$ will be invariant if and only if the prolongation of X applied to the radial Euler equations (2.1)–(2.3) vanishes when evaluated on $\mathcal{E}$.

A simpler, modern formulation of invariance [19, 9, 5] comes from using the characteristic form of generator and is given by the condition that the Frechet derivative of the equations (2.1)–(2.3) must vanish on $\mathcal{E}$:

$$\big(D_tP^U + UD_rP^U + U_rP^U + D_r(p_SP^S + p_\rho P^\rho)/\rho - (p_SS_r + p_\rho\rho_r)P^\rho/\rho^2\big)\big|_{\mathcal{E}} = 0, \tag{3.5a}$$

$$\big(D_tP^\rho + D_r(UP^\rho + \rho P^U) + \tfrac{n-1}{r}(UP^\rho + \rho P^U)\big)\big|_{\mathcal{E}} = 0, \tag{3.5b}$$

$$\big(D_tP^S + UD_rP^S + S_rP^U\big)\big|_{\mathcal{E}} = 0. \tag{3.5c}$$

These determining equations split with respect to derivatives of (U, ρ, S), yielding an overdetermined system of PDEs which can be solved for $\tau, \xi, \eta^U, \eta^\rho, \eta^S$ along with $p(\rho, S) \neq$const. and $n \neq 1$. This gives the following classification result. Remarks on the computation are given in Appendix A.

Theorem 3.1. *(i) For a general equation of state* (2.4)*, the Lie point symmetries are generated by a time-translation ∂_t and a dilation $t\partial_t + r\partial_r$. (ii) Additional Lie point symmetries exist only for the equations of state shown in Table 1 and their specializations, modulo an additive constant. (iii) A classification of all admitted maximal point-symmetry algebras is shown in Table 2.*

p	# extra symmetries
$\kappa(S)f(\rho)$	1
$f(\rho) + \kappa(S)$	1
$f(\kappa(S)\rho)\rho^{1+q}$, $q \neq -1$	1
$f(\kappa(S)\rho) + k\ln\rho$	1
$f(\rho)$	1
$\kappa(S)\rho^{1+2/n}$	1
$\kappa(S)$	1

Table 1. Equations of state of maximal generality admitting extra Lie point symmetries. f and κ are non-constant functions.

The cases in Table 1 are organized by generality of the equation of state. In each case, the number of extra symmetries counts only those symmetries that do not arise by a linear combination of symmetries inherited from intersections of more general cases.

In Table 2, the cases are organized by dimension and arise from specializations of the equations of state listed in Table 1. The resulting classification is complete in the sense that every point symmetry admitted for any given equation of state appears among a linear combination of the listed symmetries or special cases of them. The notation for the algebras is taken from Ref. [20].

Note that the equations of state in cases 6 and 7 in Table 2 do not appear in Table 1. Their symmetries arise as linear combinations of the symmetries inherited from more general cases. Specifically, case 6 is given by the intersection of case 2 with $f = \rho^{1+q}$ and case 5 with $f = (\kappa(S)^{\frac{1}{q+1}}\rho)^{1+q}$, $k = 0$, whereby $\mathrm{X}_7 = (q+1)\mathrm{X}_3|_{f=\rho^{1+q}} - \mathrm{X}_6|_{f=(\kappa(S)^{\frac{1}{q+1}}\rho)^{1+q},k=0}$. Likewise, case 7 is given by the intersection of case 3 with $f = k\ln\rho$ and case 4 with $f = k\ln(e^{\kappa(S)/k}\rho)$, $q = -1$, whereby $\mathrm{X}_8 = -2k\mathrm{X}_4|_{f=k\ln\rho} - \mathrm{X}_5|_{f=k\ln(e^{\kappa(S)/k}\rho),q=-1}$.

case	p	generators & non-zero commutators	algebra
1	$f(\rho, S)$	$\mathrm{X}_1 = \partial_t$, $\mathrm{X}_2 = t\partial_t + r\partial_r$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1}$
2	$\kappa(S)f(\rho)$	X_1, X_2, $\mathrm{X}_3 = r\partial_r + U\partial_U + \frac{2\kappa(S)}{\kappa'(S)}\partial_S$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus \mathrm{A}_1$
3	$\kappa(S) + f(\rho)$	X_1, X_2, $\mathrm{X}_4 = \frac{1}{\kappa'(S)}\partial_S$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus \mathrm{A}_1$
4	$f(\kappa(S)\rho)\rho^{1+q}$ $q \neq -1$	X_1, X_2, $\mathrm{X}_5 = qr\partial_r + qU\partial_U + 2\rho\partial_\rho - \frac{2\kappa(S)}{\kappa'(S)}\partial_S$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus \mathrm{A}_1$
5	$f(\kappa(S)\rho) + k\ln\rho$	X_1, X_2, $\mathrm{X}_6 = r\partial_r + U\partial_U - 2\rho\partial_\rho + \frac{2\kappa(S)}{\kappa'(S)}\partial_S$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus \mathrm{A}_1$
6	$\kappa(S)\rho^{1+q}$ $q \neq -1$	X_1, X_2, X_3, $\mathrm{X}_7 = qr\partial_r + qU\partial_U + 2\rho\partial_\rho$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus 2\mathrm{A}_1$
7	$\kappa(S) + k\ln\rho$ $k \neq 0$	X_1, X_2, X_4, $\mathrm{X}_8 = r\partial_r + U\partial_U - 2\rho\partial_\rho$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus 2\mathrm{A}_1$
8	$\kappa(S)\rho^{1+2/n}$	X_1, X_2, X_3, $\mathrm{X}'_7 = r\partial_r + U\partial_U + n\rho\partial_\rho$, $\mathrm{X}_9 = t^2\partial_t + r\,t\partial_r + (r - tU)\partial_U - nt\rho\partial_\rho$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$, $[\mathrm{X}_1, \mathrm{X}_9] = 2\mathrm{X}_2 - \mathrm{X}'_7$, $[\mathrm{X}_2, \mathrm{X}_9] = \mathrm{X}_9$	$\mathfrak{sl}(2, \mathbb{R}) \oplus 2\mathrm{A}_1$
9	$f(\rho)$ $(\rho f')' \neq 0$ $(\rho f'/f)' \neq 0$	X_1, X_2, $\mathrm{X}_{10} = F(S)\partial_S$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus \mathrm{A}_\infty$
10	$\kappa(S)$	X_1, X_2, X_8, $\mathrm{X}_{11} = ((F(S)\kappa'(S))'/\kappa'(S))\rho\partial_\rho + F(S)\partial_S$ $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus \mathrm{A}_1 \oplus \mathrm{A}_\infty$
11	$k\ln\rho$ $k \neq 0$	X_1, X_2, X_8, X_{10} $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus \mathrm{A}_1 \oplus \mathrm{A}_\infty$
12	$k\rho^{1+q}$ $k \neq 0$, $q \neq -1$	X_1, X_2, X_7, X_{10} $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$	$\mathrm{A}_{2,1} \oplus \mathrm{A}_1 \oplus \mathrm{A}_\infty$
13	$k\rho^{1+2/n}$ $k \neq 0$	X_1, X_2, X_9, X'_7, X_{10} $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_1$, $[\mathrm{X}_1, \mathrm{X}_9] = 2\mathrm{X}_2 - \mathrm{X}'_7$, $[\mathrm{X}_2, \mathrm{X}_9] = \mathrm{X}_9$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathrm{A}_1 \oplus \mathrm{A}_\infty$

TABLE 2. Maximal Lie point symmetry algebras.
f and κ are non-constant functions; F is a non-zero function.

Similarly, the equations of state in cases 11, 12, and 13 do not appear in Table 1, because all of the symmetries in these cases are directly inherited from the intersection of case 9 with cases 7, 6, 8, respectively.

Also note that case 8 contains a specialization of case 6, where $\mathrm{X}'_7 = \frac{1}{q}\mathrm{X}_7\big|_{q=\frac{2}{n}}$.

Remark: For each case in Table 2, it is straightforward to derive a system of differential equations and inequations that involve only p and n, whose general solution yields p. Such a characterization of cases is useful for determining which case contains a given equation of state, by checking which system the equation of state satisfies.

To conclude this discussion of Lie point symmetries, the transformation groups generated by each symmetry will now be presented.

Proposition 3.2. *The infinitesimal symmetries listed in Table 2 generate the following transformation groups (with ϵ being the group parameter):*

$$X_1 : t \to t + \epsilon; \quad \textit{time translation} \tag{3.6}$$

$$X_2 : t \to e^{\epsilon} t, \quad r \to e^{\epsilon} r; \quad \textit{dilation} \tag{3.7}$$

$$X_5 : r \to e^{q\epsilon} r, \quad U \to e^{q\epsilon} U, \quad \rho \to e^{2\epsilon}\rho, \quad S \to \kappa^{-1}(e^{-2\epsilon}\kappa(S)); \quad \textit{combined scaling \& entropy shift} \tag{3.8}$$

$$X_8 : r \to e^{\epsilon} r, \quad U \to e^{\epsilon} U, \quad \rho \to e^{-2\epsilon}\rho, \quad S \to \kappa^{-1}(e^{2\epsilon}\kappa(S)); \quad \textit{combined scaling \& entropy shift} \tag{3.9}$$

$$X_3 : r \to e^{\epsilon} r, \quad U \to e^{\epsilon} U, \quad S \to \kappa^{-1}(e^{2\epsilon}\kappa(S)); \quad \textit{combined scaling \& entropy shift} \tag{3.10}$$

$$X_7 : r \to e^{q\epsilon} r, \quad U \to e^{q\epsilon} U, \quad \rho \to e^{2\epsilon}\rho; \quad \textit{scaling} \tag{3.11}$$

$$X_9 : t \to t/(1-\epsilon t), \quad r \to r/(1-\epsilon t), \quad U \to (1-\epsilon t)U + \epsilon r, \quad \rho \to (1-\epsilon t)^n \rho; \quad \textit{conformal similarity} \tag{3.12}$$

$$X_4 : S \to \kappa^{-1}(\kappa(S) + \epsilon); \quad \textit{entropy change} \tag{3.13}$$

$$X_{10} : S \to H^{-1}(H(S) + \epsilon); \quad \textit{entropy change} \tag{3.14}$$

$$X_{11} : \rho \to \rho\kappa'(H^{-1}(H(S)+\epsilon))F(H^{-1}(H(S)+\epsilon))/(\kappa'(S)H(S)), \quad S \to H^{-1}(H(S)+\epsilon); \quad \textit{entropy change} \tag{3.15}$$

where $H'(y) = 1/F(y)$.

Some remarks are worthwhile. An invariant of the groups (3.8) and (3.9) is $\kappa(S)\rho$. The group (3.12) acts as a conformal scaling on t, r, $U - t/r$, ρ, with a scaling factor $1 - \epsilon t$, where the scaling weight of t, r, $U - t/r$ is 1, and the scaling weight of ρ is n. The group (3.13) corresponds to $\kappa(S) \to \kappa(S) + \epsilon$. Likewise, the groups (3.14) and (3.15) have $H(S) \to H(S) + \epsilon$, while $\kappa'(S)F(S)/\rho$ is an invariant of the latter.

4. Casimirs

A Casimir is a non-trivial functional $C = \int_{V(t)} \Phi^t \, r^{n-1} dr$ such that

$$\{C, F\} = 0 \tag{4.1}$$

(modulo a trivial functional) for all functionals F on $V(t)$. Existence of a Casimir indicates that the Poisson bracket associated to $\mathcal{H}$ is degenerate. If Φ^t has no explicit dependence on t, then C is a conserved integral, as a consequence of relation (2.15). A symmetry characterization of Casimirs is stated in the next section.

From the definition (2.14) of the Poisson bracket, the condition (4.1) is equivalent to $\mathcal{H}\nabla_{U,\rho,S} C = 0$. This can be used as a determining system to find the Casimirs of the radial Euler equations (2.1)–(2.3). After simplification, the determining system is given by

$$E_U(r^{n-1}\Phi^t) = 0, \quad D_r(r^{1-n}E_\rho(r^{n-1}\Phi^t)) = r^{1-n}(S_r/\rho)E_S(r^{n-1}\Phi^t) \tag{4.2}$$

in terms of the Euler operators E_v with respect to $v = (U, \rho, S)$. It is computationally straightforward to determine all Casimirs given by conserved densities $\Phi^t(r, U, \rho, S, U_r, \rho_r, S_r)$ up to first order, modulo trivial conserved densities.

Proposition 4.1. *For a general equation of state for the radial Euler equations* (2.1)–(2.3)*, Casimirs having a conserved density up to first order are given by* $\Phi^t = \rho f(J_0, J_1)$*, where* $J_0 = S$ *and* $J_1 = r^{1-n}S_r/\rho$ *are the lowest-order invariants in the hierarchy* (2.27)*. No other first-order Casimirs exist for special equations of state.*

Since all first-order Casimirs $C = \int_{V(t)} \rho f(J_0, J_1)\, r^{n-1}dr$ coincide with the advected integrals $\mathcal{I}_1$ of order 1, a natural question is whether the entire hierarchy of these integrals (2.26) are Casimirs.

Theorem 4.2. *For a general equation of state* (2.4)*, every advected integral* (2.26) *is a Casimir.*

The proof will be given in the next subsection.

In contrast, in the other hierarchy of advected integrals (2.29), which hold only for entropic equations of state, none are Casimirs apart from the ones that also belong to the first hierarchy. This result is easily seen from the first equation in the determining system (4.2), which implies that the conserved density Φ^t in a Casimir must have no essential dependence on U. Thus, all of the advected integrals involving at least one of $J_{1,l}$, $J_{2,l}$, $l = 1, 2, \ldots$, cannot be Casimirs since they depend explicitly on U (and its r-derivatives).

The question of whether there exist any additional Casimirs is a much harder problem which will be left for elsewhere.

4.1. **Proof of Theorem 4.2.** The determining system (4.2) can be expressed in the form

$$E_U(\tilde{\Phi}) = 0, \quad J_1 E_S(\tilde{\Phi}) = D_r E_{\tilde{\rho}}(\tilde{\Phi}) \tag{4.3}$$

where $\tilde{\rho} = r^{n-1}\rho$, and $\tilde{\Phi} = r^{n-1}\Phi^t$. Consider, hereafter, $\Phi^t = \rho f(J_l)$.

The proof is by induction. Since $J_0 = S$, then $\tilde{\Phi} = r^{n-1}\rho f(J_0) = \tilde{\rho} f(S)$, which is the density in the advected generalized-entropy integral (2.20). Hence, equations (4.3) hold for $\tilde{\Phi} = \tilde{\rho} f(J_0)$.

Now suppose that the determining equations (4.3) hold for $\tilde{\Phi} = \tilde{\rho} f(J_k)$, $k \geq 0$. The induction step requires showing that the determining equations then hold for $\tilde{\Phi} = \tilde{\rho} f(J_{k+1})$. This will be accomplished by splitting the equations with respect to derivatives of f, which yields an equivalent system formulated in terms of Euler-Lagrange operators applied to the invariant J_l as follows.

Lemma 4.3. *The Casimir determining equations* (4.3) *are equivalent to the split system*

$$J_1 E_S(J_k) = \mathcal{R} J_k + D_r E_{\tilde{\rho}}(J_k), \tag{4.4a}$$

$$J_1 E_S^{(i)}(J_k) = D_r E_{\tilde{\rho}}^{(i)}(J_k) - E_{\tilde{\rho}}^{(i-1)}(J_k), \quad i = 1, 2, \ldots. \tag{4.4b}$$

To derive this system (4.4) from the two determining equations (4.3), observe that the first determining equation holds identically, since J_k has no dependence on U and its derivatives. Next, the second determining equation can be expressed in terms of derivatives of f by use of the relation

$$J_1 \sum_{i\geq 0} (-D_r)^i (\tilde{\rho} f'(J_k)) E_S^{(i)}(J_k) = \tilde{\rho} f'(J_k)\big(\mathcal{R} J_k + D_r E_{\tilde{\rho}}(J_k)\big) \\ + \sum_{i\geq 1} (-D_r)^i (\tilde{\rho} f'(J_k))\big(D_r E_{\tilde{\rho}}^{(i)}(J_k) - E_{\tilde{\rho}}^{(i-1)}(J_k)\big). \tag{4.5}$$

(This relation holds by Euler-operator identity (B.2), with $a = \tilde{\rho}$ and $b = J_k$.) Because f is an arbitrary function of J_k, the coefficients of $(-D_r)^i(\tilde{\rho} f'(J_k))$ for $i = 0, 1, 2, \ldots$ on each side of equation (4.5) must be equal. As a result, this equation splits into the system (4.4).

The proof of the induction step now starts from the determining equations (4.3) with $\tilde{\Phi} = \tilde{\rho} f(J_{k+1})$, where $J_{k+1} = \mathcal{R} J_k = (D_r J_k)/\tilde{\rho}$ via the recursion operator (2.28). The first equation holds identically, since J_k has no dependence on U and its derivatives. The second equation splits similarly to the system (4.4) by use of the relation

$$J_1 \sum_{i\geq 0} ((-D_r)^{i+1} f'(J_{k+1})) E_S^{(i)}(J_k) = -D_r f'(J_{k+1})\big(J_{k+1} + D_r E_z(J_k)\big) \\ + \sum_{i\geq 1} ((-D_r)^{i+1} f'(J_{k+1}))\big(D_r E_{\tilde{\rho}}^{(i)}(J_k) - E_{\tilde{\rho}}^{(i-1)}(J_k)\big) \tag{4.6}$$

(which holds by the Euler-operator identity (B.3), with $a = \tilde{\rho}$ and $b = J_k$). It is easy to see that the coefficients of $(-D_r)^{i+1} f'(J_{k+1}))$ for $i = 0, 1, 2, \ldots$ on each side of this equation are equal due to the split system (4.4). Hence, the determining equations (4.3) hold for $\tilde{\Phi} = \tilde{\rho} f(J_{k+1})$. This establishes the induction.

The preceding argument can be extended straightforwardly to the general case where f depends on all invariants $J_0, J_1, \ldots, J_l$. This completes the proof of Theorem 4.2.

5. Hamiltonian symmetries

A generalization of infinitesimal Lie point symmetries arises from allowing the components in the generator (3.1) to depend additionally on derivatives of U, ρ, S, such that the determining equations (3.5) hold using the characteristic form (3.4) of the generator. If its components (P^U, P^ρ, P^S) involve derivatives up to order $k \geq 1$, then such a generator is called a symmetry of order k, or sometimes, a generalized symmetry.

Any generalized symmetry in characteristic form (3.4) can be expressed in an equivalent form (3.1) where τ and ξ are any functions of t, r, U, ρ, S, and their derivatives, while η^U, η^ρ, η^S are given by the relations (3.3) in terms of (P^U, P^ρ, P^S). Lie point symmetries are characterized by the property that there is a unique choice of τ and ξ depending only on t, r, U, ρ, S, for which η^U, η^ρ, η^S have no dependence on derivatives of U, ρ, S.

One main property of the Hamiltonian structure (2.11)–(2.13) is that, for any conserved integral $G = \int_{V(t)} \Phi^t\, r^{n-1} dr$, the action of the Hamiltonian operator $\mathcal{H}$ yields a generalized symmetry (3.4) whose components are given by

$$\big(P^U,\ P^\rho,\ P^S\big)^{\mathrm{t}} = \mathcal{H} \nabla_{U,\rho,S} G, \tag{5.1}$$

namely

$$
\begin{aligned}
P^U &= -D_r(r^{1-n}E_\rho(r^{n-1}\Phi^t)) + r^{1-n}(S_r/\rho)E_S(r^{n-1}\Phi^t),\\
P^\rho &= -r^{1-n}D_rE_U(r^{n-1}\Phi^t),\\
P^S &= -r^{1-n}(S_r/\rho)E_U(r^{n-1}\Phi^t).
\end{aligned}
\tag{5.2}
$$

A symmetry of this form is called a Hamiltonian symmetry.

A general result in Hamiltonian theory states that if F and G are two conserved integrals, then the commutator of the corresponding Hamiltonian symmetries is a Hamiltonian symmetry corresponding to the Poisson bracket (2.14) of F and G. Thus, the Poisson bracket algebra of a closed set of conserved integrals is isomorphic to the Lie algebra of the corresponding set of Hamiltonian symmetries. If a conserved integral yields a Hamiltonian symmetry which is trivial, $\hat{\mathrm{X}} \equiv 0$, then it is a Casimir. In general, Hamiltonian symmetries may not exhaust all of the symmetries admitted by the Hamiltonian equations of motion.

The Hamiltonian symmetries arising from all of the kinematic conserved integrals (2.19)–(2.25) possessed by the radial Euler equations (2.1)–(2.3) will now be derived from expressions (5.2).

The mass and generalized-entropy integrals (2.19)–(2.20) yield a trivial symmetry $\hat{\mathrm{X}} = 0$, since they are special cases of the Casimir $C = \mathcal{I}_0 = \int_{V(t)} \rho f(J_0)\, r^{n-1}dr$ with f being an arbitrary function of $J_0 = S$.

For the remaining 5 kinematics conserved integrals (2.21)–(2.25), the Hamiltonian symmetries are shown in Table 3. For each one, a suitable choice of τ and ξ is made so that the symmetry generator takes the simplest possible form. The respective choices are given by the coefficients of D_t and D_r in the expressions for (P^U, P^ρ, P^S) in terms of (U, ρ, S).

As expected, the energy integral (2.21) corresponds to time-translation symmetry. The dilational and similarity energies (2.22) and (2.23) correspond to scaling and conformal similarity symmetries, which are Lie point symmetries. The resulting transformation groups generated by these symmetries are shown in section 3.

In contrast, both the enthalpy flux integral (2.24) and the entropy-weighted energy integral (2.25) correspond to first-order generalized symmetries. Each of these symmetries generates a transformation group acting on solutions of the radial Euler equations (2.1)–(2.3), where the group is defined by the system of first-order PDEs for $(U^*(r,t;\epsilon), \rho^*(r,t;\epsilon), S^*(r,t;\epsilon))$,

$$
U^*_\epsilon = P^U|_{(U^*,\rho^*,S^*)},\quad \rho^*_\epsilon = P^\rho|_{(U^*,\rho^*,S^*)},\quad S^*_\epsilon = P^S|_{(U^*,\rho^*,S^*)}
\tag{5.3}
$$

in terms of the components of the symmetry generator, with ϵ denoting the group parameter such that $(U^*(r,t;0), \rho^*(r,t;0), S^*(r,t;0)) = (U(r,t), \rho(r,t), S(r,t))$.

For the symmetry arising from the enthalpy flux integral (2.24), the determining system (5.3) is given by

$$
U^*_\epsilon = 0,\quad \rho^*_\epsilon = 0,\quad S^*_\epsilon + r^{1-n}S^*_r/\rho^* = 0.
\tag{5.4}
$$

Explicit integration of this system yields the transformation group

$$
U^* = U(r,t),\quad \rho^* = \rho(r,t),\quad S^* = S(M^{-1}(M(r,t) - \epsilon), t)
\tag{5.5}
$$

where the function $M(r) = \int_0^r \rho(r,t)\, r^{n-1}dr$ is the mass contained in the radial domain $[0,r]$ at any fixed time t, and M^{-1} denotes the inverse function. Similarly, the determining system (5.3) defined by the symmetry arising from the entropy-weighted energy integral

(2.25) consists of

$$U^*_\epsilon + f(S^*)U^*_t = 0, \quad \rho^*_\epsilon + f(S^*)\rho^*_t - f'(S^*)S^*_r\rho^* = 0, \quad S^*_\epsilon + f(S^*)S^*_t = 0. \tag{5.6}$$

This first-order system can be integrated to obtain the transformation group given by

$$U^* = U(r,\sigma), \quad \rho^* = \rho(r,\sigma)\big(1+\epsilon S_t(r,\sigma)f'(S^*)\big)^{S_r(r,\sigma)/S_t(r,\sigma)}, \tag{5.7}$$

with $S^* = S^*(r,t;\epsilon)$ being the implicit solution of

$$S^* = S(r,\sigma), \quad \sigma = t - \epsilon f(S^*). \tag{5.8}$$

conserved integral	p	(P^U, P^ρ, P^S)	symmetry
energy (2.21)	$f(\rho,S)$	$-D_t(U,\rho,S)$	time-translation X_1
dilational energy (2.22)	$\kappa(S)\rho^{1+q}$	$(U, \frac{2}{q}\rho, 0) - rD_r(U,\rho,S)$	scaling X_7
similarity energy (2.23)	$\kappa(S)\rho^{1+q}$	$(r-tU, -nt\rho, 0) - (t^2D_t + rtD_r)(U,\rho,S)$	conformal similarity X_9
enthalpy flux (2.24)	$f(\rho)$	$(0,0,-J_1)$	1st-order $X = -J_1\partial_S$
entropy-weighted energy (2.25)	$\kappa(S)$	$-f(S)D_t(U,\rho,S) + (0, f'(S)S_r\rho, 0)$	1st-order $X = f(S)\partial_t + f'(S)S_r\rho\partial_\rho$

TABLE 3. Hamiltonian symmetries from kinematic conserved integrals

In addition to kinematic conserved integrals, the hierarchy of non-kinematic advected integrals (2.29) for entropic equations of state yield non-trivial Hamiltonian symmetries since, as shown in the previous section, none of them are Casimirs. The resulting symmetries are more complicated in comparison to the symmetries in Table 3.

Firstly, the two simplest advected integrals will be considered:

$$\mathcal{I}'_1\big|_{f=J_{1,1}} = \int_{V(t)} \rho J_{1,1}\, r^{n-1}dr, \tag{5.9}$$

$$\mathcal{I}'_1\big|_{f=J_{2,1}} = \int_{V(t)} \rho J_{2,1}\, r^{n-1}dr, \tag{5.10}$$

which involve the advected scalars $J_{1,1} = U^2 + \frac{2}{n}rp_r/\rho$ and $J_{2,1} = A(r,U,p_r/\rho) - t$ with $A(r,U,p_r/\rho)$ given by expression (2.32).

Recall that, as remarked in section 2, the advected integral (5.9) is equivalent to the energy integral (2.21) (modulo a trivial conserved integral) specialized to the case of an entropic equation of state. For this conserved integral, the correspondence (5.2) yields $(P^U, P^\rho, P^S) = 2D_t(U,\rho,S)$ which is equivalent to a time-translation symmetry

$$X_{J_{1,1}} = -2X_1. \tag{5.11}$$

For the other advected integral (5.10), the correspondence (5.2) gives $(P^U, P^\rho, P^S) = -(J_{1,2})_U D_r(U,\rho,S) - \big((J_{1,2})_r, \rho(D_r(J_{1,2})_U + \frac{n-1}{r}(J_{1,2})_U), 0\big)$. This yields the 2nd-order symmetry

$$X_{J_{2,1}} = A_U\partial_r - A_r\partial_U - \rho(D_rA_U + \tfrac{n-1}{r}A_U)\partial_\rho. \tag{5.12}$$

The two symmetries (5.11) and (5.12) commute, namely $[\mathrm{pr}\hat{\mathrm{X}}_{J_{1,1}}, \mathrm{pr}\hat{\mathrm{X}}_{J_{2,1}}] = 0$ using their characteristic form (3.4), where pr denotes prolongation.

Similar results hold for the whole hierarchy of advected integrals (2.29). A proof is provided in the next subsection.

Theorem 5.1. *The advected integral*

$$\mathcal{I}'_l\big|_{f(J_{1,l},J_{2,l})} = \int_{V(t)} \rho f(J_{1,l}, J_{2,l})\, r^{n-1} dr, \tag{5.13}$$

for any $l \geq 1$, yields the lth-order Hamiltonian symmetry

$$\mathrm{X}_{f(J_{1,l},J_{2,l})} = -2f_{1,l}^{(l-1)}\partial_t + A_U f_{2,l}^{(l-1)}\partial_r - A_r f_{2,l}^{(l-1)}\partial_U + \big(2D_t f_{1,l}^{(l-1)} - D_r(A_U f_{2,l}^{(l-1)}) - \tfrac{n-1}{r} A_U f_{2,l}^{(l-1)}\big)\rho\partial_\rho \tag{5.14}$$

where $f_{\cdot,l}^{(i)} = (-\mathcal{R})^i f_{J_{\cdot,l}}$, with $\mathcal{R}$ being the recursion operator (2.28)*.*

The set of all symmetries (5.14) comprises a Lie algebra which has a non-trivial commutator structure. As an illustration, consider the lowest-order case $l = 1$. Then a direct computation shows that

$$[\mathrm{pr}\hat{\mathrm{X}}_{f(J_{1,1},J_{2,1})}, \mathrm{pr}\hat{\mathrm{X}}_{g(J_{1,1},J_{2,1})}] = \mathrm{pr}\hat{\mathrm{X}}_{h(J_{1,1},J_{2,1})}, \quad h = 2f_{J_{1,1}}g_{J_{2,1}} - 2f_{J_{2,1}}g_{J_{1,1}}. \tag{5.15}$$

The symmetry $\mathrm{X}_{h(J_{1,1},J_{2,1})}$ here will be non-trivial if and only if h is not linear in $(J_{1,1}, J_{2,1})$. For instance, if f and g are polynomials, then at least one of them must be at least quadratic, otherwise $\mathrm{X}_{h(J_{1,1},J_{2,1})}$ will vanish. A similar result holds for $l \geq 2$.

Thus, the radial Euler equations (2.1)–(2.3) possess a rich structure of Hamiltonian symmetries.

5.1. **Proof of Theorem 5.1.** The first step will use the following result for evaluating the Euler operator applied to the densities in the two integrals (5.13).

Lemma 5.2. *Let $K = K(r, U, \rho, S, S_r)$ and $f(K)$ be arbitrary (smooth) functions of their arguments. Denote $K_i = \mathcal{R}^i K$ and $f_K^{(i)} = (-\mathcal{R})^i f'(K)$, $i \geq 0$, using the recursion operator* (2.28)*. Then:*

$$E_U(r^{n-1}\rho f(K_l)) = f_K^{(l)} E_U(r^{n-1}\rho K), \tag{5.16}$$

$$E_S(r^{n-1}\rho f(K_l)) = f_K^{(l)} E_S(r^{n-1}\rho K) - (D_r f_K^{(l)}) E_S^{(1)}(r^{n-1}\rho K), \tag{5.17}$$

$$E_\rho(r^{n-1}\rho f(K_l)) = f_K^{(l)} E_\rho(r^{n-1}\rho K) + r^{n-1}\big(f - \mathcal{D}_l f'\big), \tag{5.18}$$

where $\mathcal{D}_l = \sum_{0\leq i\leq l} K_{l-i}(-\mathcal{R})^i$. Moreover:

$$D_r\big(f(K_l) - \mathcal{D}_l f'(K_l)\big) = -K D_r f_K^{(l)}. \tag{5.19}$$

These identities (5.16)–(5.18) can be derived by the following descent argument. Firstly, consider the lefthand side of (5.16), and successively apply the Euler-operator identities (B.3) and (B.4) using $v = U$, $a = r^{n-1}\rho$ and $b = K_l$. This yields: $E_U(r^{n-1}\rho f(K_l)) = \sum_{1\leq i\leq l} E_U^{(i-1)}(K_{l-1})(-D_r)^i f' = -\sum_{1\leq i\leq l-1} E_U^{(i-1)}(K_{l-2})(-D_r)^i f_K^{(1)}$. Iteration of this step leads to the righthand side of (5.16), using the property that $E_U^i(K) = 0$ for $i \geq 1$ since K does not depend on derivatives of U. Secondly, consider the lefthand side of (5.17). The previous steps with $v = \rho$ lead to the righthand side of (5.17) where the additional term arises because K depends on S_r (but not any higher derivatives of S). Thirdly, the derivation of (5.18) is similar and uses properties that $E^i_\rho(K) = 0$ and $E^i_\rho(a) = 0$ for $i \geq 1$, as well as $E_\rho(a) = r^{n-1}$.

Now, returning to the main proof, the respective cases $f = f(J_{1,1})$ and $f = f(J_{2,1})$ will be considered first. Both cases involve the same steps.

Case $f = f(J_{1,1})$: Put $K = J_{1,1}$ into the identities (5.16)–(5.18) where $r^{n-1}\rho K = r^{n-1}\rho U^2 + \frac{2}{n}(r/\rho)p_r$. Use of

$$\begin{aligned} E_U(r^{n-1}\rho J_{1,1}) = 2r^{n-1}\rho U, \quad E_\rho(r^{n-1}\rho J_{1,1}) = 2r^{n-1}U^2, \\ E_S(r^{n-1}\rho J_{1,1}) = -2r^{n-1}p', \quad E^{(1)}_S(r^{n-1}\rho J_{1,1}) = \tfrac{2}{n}r^n p' \end{aligned} \tag{5.20}$$

leads to the expressions

$$Q^U_{1,l} \equiv E_U(r^{n-1}\rho f(J_{1,l})) = 2r^{n-1}\rho U f^{(l)}_{J_{1,l}}, \tag{5.21}$$

$$Q^\rho_{1,l} \equiv E_\rho(r^{n-1}\rho f(J_{1,l})) = r^{n-1}\big(U^2 f^{(l)}_{J_{1,l}} + f - \sum_{0\leq i\leq l} J_{1,l-i} f^{(i)}_{J_{1,l}}\big), \tag{5.22}$$

$$Q^S_{1,l} \equiv E_S(r^{n-1}\rho f(J_{1,l})) = -\tfrac{2}{n}p' D_r(r^n f^{(l)}_{J_{1,l}}). \tag{5.23}$$

The next step in the proof of this case is to substitute expressions (5.21)–(5.23) into the correspondence (5.2) to obtain the components (P^U, P^ρ, P^S) of the Hamiltonian symmetry. Proceeding in order of simplicity: first,

$$P^S = -(r^{1-n}S_r/\rho)Q^U_{1,l} = -2US_r f^{(l)}_{J_{1,l}}; \tag{5.24}$$

second,

$$P^\rho = -r^{1-n}D_r Q^U_{1,l} = -2((\rho U)_r + \tfrac{n-1}{r}\rho U) f^{(l)}_{J_{1,l}} - 2\rho U D_r f^{(l)}_{J_{1,l}}; \tag{5.25}$$

and last,

$$\begin{aligned} P^U &= (r^{1-n}S_r/\rho)Q^S_{1,l} - D_r(r^{1-n}Q^\rho_{1,l}) \\ &= -\tfrac{2}{n}(r^{1-n}p_r/\rho)D_r(r^n f^{(l)}_{J_{1,l}}) - D_r(U^2 f^{(l)}_{J_{1,l}} + f - \mathcal{D}_l f'(J_{1,l})) \\ &= -2(UU_r + p_r/\rho) f^{(l)}_{J_{1,l}} \end{aligned} \tag{5.26}$$

which uses $D_r(f - \mathcal{D}_l f'(J_{1,l})) = -J_{1,l}D_r f^{(l)}_{J_{1,l}}$ via identity (5.19).

For the final step in the proof of the first case, the r-derivatives in the expressions (5.24)–(5.26) can replaced in terms of t-derivatives through the radial Euler equations (2.1)–(2.3). Likewise, since $J_{1,l}$ is an advected invariant, it satisfies $D_t J_{1,l} + UD_r J_{1,l} = 0$, which implies $UD_r f^{(l)}_K(J_{1,l}) = -D_t f^{(l)}_K(J_{1,l})$ since $\mathcal{R}$ is a recursion operator on advected invariants. Hence, this yields

$$P^S = 2S_t f^{(l)}_{J_{1,l}}, \quad P^\rho = 2D_t(\rho f^{(l)}_{J_{1,l}}), \quad P^U = 2U_t f^{(l)}_{J_{1,l}}, \tag{5.27}$$

which can be expressed more simply as

$$(P^U, P^\rho, P^S) = 2f^{(l)}_{J_{1,l}} D_t(U, \rho, S) + 2(0, \rho D_t f^{(l)}_{J_{1,l}}, 0), \tag{5.28}$$

giving the Hamiltonian symmetry

$$\mathrm{X}_{f(J_{1,l})} = -2f^{(l-1)}_{1,l}\partial_t + (2D_t f^{(l-1)}_{1,l})\rho\partial_\rho. \tag{5.29}$$

Case $f = f(J_{2,1})$: Put $K = J_{2,1}$ into the identities (5.16)–(5.18) where $r^{n-1}\rho K = r^{n-1}\rho(A(r,U,p_r/\rho) - t)$. Use of

$$\begin{aligned} E_U(r^{n-1}\rho J_{2,1}) = r^{n-1}\rho A_U, \quad E_\rho(r^{n-1}\rho J_{2,1}) = r^{n-1}(J_{2,1} - p_r A_{p_r}), \\ E_S(r^{n-1}\rho J_{2,1}) = -\kappa'(S) D_r(r^{n-1}\rho A_{p_r}), \quad E_S^{(1)}(r^{n-1}\rho J_{2,1}) = r^{n-1}\rho\kappa'(S) A_{p_r} \end{aligned} \tag{5.30}$$

leads to the expressions

$$Q^U_{2,l} \equiv E_U(r^{n-1}\rho f(J_{2,l})) = r^{n-1}\rho A_U f^{(l)}_{J_{2,l}}, \tag{5.31}$$

$$Q^\rho_{2,l} \equiv E_\rho(r^{n-1}\rho f(J_{2,l})) = r^{n-1}\big(f - p_r A_{p_r} f^{(l)}_{J_{2,l}} - \sum_{1\leq i\leq l} J_{2,l-i} f^{(i)}_{J_{2,l}}\big), \tag{5.32}$$

$$Q^S_{2,l} \equiv E_S(r^{n-1}\rho f(J_{2,l})) = -\kappa'(S) D_r(r^{n-1}\rho A_{p_r} f^{(l)}_{J_{2,l}}). \tag{5.33}$$

The components (P^U, P^ρ, P^S) of the Hamiltonian symmetry are then given by

$$P^S = -(r^{1-n}S_r/\rho) Q^U_{2,l} = -S_r A_U f^{(l)}_{J_{2,l}}, \tag{5.34}$$

$$P^\rho = -r^{1-n} D_r Q^U_{1,l} = -D_r(\rho A_U f^{(l)}_{J_{2,l}}) - \tfrac{n-1}{r}\rho A_U f^{(l)}_{J_{2,l}}, \tag{5.35}$$

$$P^U = (r^{1-n}S_r/\rho) Q^S_{1,l} - D_r(r^{1-n}Q^\rho_{1,l}) = -(A_r + U_r A_U) f^{(l)}_{J_{2,l}} \tag{5.36}$$

similarly to the previous case. Hence,

$$(P^U, P^\rho, P^S) = -A_U f^{(l)}_{J_{2,l}} D_r(U,\rho,S) - \big(A_r f^{(l)}_{J_{2,l}}, (D_r(A_U f^{(l)}_{J_{2,l}}) + \tfrac{n-1}{r}A_U)\rho, 0\big) \tag{5.37}$$

gives the Hamiltonian symmetry

$$\mathrm{X}_{f(J_{2,l})} = A_U f^{(l-1)}_{2,l}\partial_r - A_r f^{(l-1)}_{2,l}\partial_U - (D_r(A_U f^{(l-1)}_{2,l}) + \tfrac{n-1}{r}A_U f^{(l-1)}_{2,l})\rho\partial_\rho. \tag{5.38}$$

Finally, the general case $f = f(J_{1,l}, J_{2,l})$ stated in Theorem 5.1 is obtained by the same steps.

It is worth remarking is that the expressions (5.21)–(5.23) and (5.31)–(5.33) are the respective multipliers that correspond to the conserved integrals $\mathcal{I}'_l\big|_{f(J_{1,l})} = \int_{V(t)} \rho f(J_{1,l})\, r^{n-1}dr$ and $\mathcal{I}'_l\big|_{f(J_{2,l})} = \int_{V(t)} \rho f(J_{2,l})\, r^{n-1}dr$, as shown by general results for conserved integrals of evolution equations (see e.g. [19, 9, 5]).

6. Concluding remarks

Radial fluid flow in $n > 1$ dimensions possesses, unexpectedly, a rich structure of point symmetries and generalized symmetries. The point symmetries comprise time-translation, space-time dilation, scaling, conformal similarity, which are well known for non-radial flow in three dimensions [13], as well as an entropy shift combined with various scalings, and an entropy change. These symmetries are found to hold for several different types of equations of state.

The time-translation, scaling, and conformal similarity are also Hamiltonian symmetries, which arise from the kinematic conserved integrals for energy, dilational energy, and similarity energy, respectively. The two "hidden" kinematic conserved integrals, describing an enthalpy-flux quantity (2.24) which holds for barotropic equations of state, and an entropy-weighted energy (2.25) which holds for entropic equations of state, give rise to first-order generalized symmetries. Each of these symmetries are shown to produce a transformation group acting on solutions of the equations for radial fluid flow.

The hierarchy of advected conserved integrals holding for a general equation of state are proved to be Hamiltonian Casimirs, which correspond to trivial symmetries. In contrast, the additional hierarchy of advected conserved integrals that hold only for an entropic equation of state give rise to a corresponding hierarchy of non-trivial generalized symmetries. These symmetries are not inherited from any symmetries of n-dimensional non-radial fluid flow. The first-order generalized symmetries are explicitly shown to comprise a non-abelian Lie algebra.

All of the preceding results carry over to radial gas dynamics through the well-known equivalence between the respective governing equations of n-dimensional gas dynamics and n-dimensional compressible fluid flow. Specifically, when a symmetry generator is expressed solely in terms of t, r, U, ρ, p, e, then it manifestly holds for both radial fluid flow and radial gas dynamics.

One direction for future investigation would be to generalize the results in the present paper and the preceding work in Ref. [1] to spherical flows in flat and curved manifolds [4].

Appendix A. Computational remarks for symmetries and Casimirs

The overdetermined system that arises from splitting the determining equations (3.5) for $\tau, \xi, \eta^U, \eta^\rho, \eta^S$ along with $p(\rho, S) \neq$const. and $n \neq 1$ contains 25 PDEs. Solving this system is a nonlinear problem which leads to many case distinctions in the solution process. To obtain all solutions, it is important that no cases are lost when integrability conditions are used and when differential equations are integrated. The program CRACK [**?**] is able reliably to carry out the computation, where the overdetermined system is obtained by the program LiePDE.

CRACK splits the computation repeatedly into cases whenever equations factorize, or coefficients of functions that are to be substituted may be zero, or integration of single differential equations with parameters has more than one solution branch.

Once all equations in every case have been solved, the solutions need to be merged into a complete case tree by eliminating solution cases that are contained in more general cases. In particular, CRACK may perform case splittings that are necessary to complete the computation automatically but that do not provide new symmetries or new equations of state.

It is straightforward to use LiePDE to determine if a case distinction in CRACK leads to a new solution case or not. The process consists of re-running CRACK from a call to LiePDE with all of the free constants and free functions in $p(\rho, S)$ being taken as fixed (namely, they are not to be solved for) in the input, and thereby solving only for the symmetry components (3.3). If the output contains fewer symmetries compared to the original solution case, then the case distinction that does not produce new symmetries and therefore is not necessary.

LiePDE and CRACK have several strengths that are relevant in the present computations of Lie point symmetries. No cases are lost; the worst that happens is that some consistent set of equations are left unsolved, and this occurs in only one case. Nearly all steps are done fully automatically. Any part of the computation can be done interactively if it is desired or needed. The whole computation of all cases with all integrations runs in a few seconds on a desktop computer.

The results were checked independently by using Maple similarly to the computation of conserved integrals in Ref. [1].

Appendix B. Euler operator identities

For a dependent variable v and an independent variable z,

$$E_v = \sum_{j\geq 0} (-D_z)^j \partial_{\partial_z^j v}, \quad E_v^{(i)} = \sum_{j\geq 0} \tbinom{i+j}{i} (-D_z)^j \partial_{\partial_z^{i+j} v}, i = 1, 2, \ldots \tag{B.1}$$

are, respectively, the Euler operator and the higher Euler operators [19, 5]. Note that $E_v^{(0)} = E_v$.

Let a and b be arbitrary smooth functions on the jet space of $v(z)$, namely $(r, v, v_z, v_{zz}, \ldots)$. Let f denote a smooth function.

The following three identities can be derived similarly to the product rule for the Euler operator:

$$E_v(af(b)) = \sum_{i\geq 0} E_v^{(i)}(b)(-D_z)^i(af'(b)) + E_v^{(i)}(a)(-D_z)^i f(b) \tag{B.2}$$

$$E_v(af(b_{+1})) = \sum_{i\geq 0} E_v^{(i)}(b)(-D_z)^{i+1} f'(b_{+1}) + E_v^{(i)}(a)(-D_z)^i(f(b_{+1}) - b_{+1}f'(b_{+1})) \tag{B.3}$$

$$\begin{aligned}\sum_{i\geq 0} E_v^{(i)}(b_{+1})(-D_z)^{i+1} f(b_{+1}) = & \sum_{i\geq 0} E_v^{(i+1)}(b)(-D_z)^{i+1}((D_z f'(b+1))/a) \\ & - \sum_{i\geq 0}\sum_{j\geq 0} \tbinom{i+j}{j} E_v^{(i+j)}(a)((-D_z)^j b_{+1})(-D_z)^i((D_z f'(b_{+1}))/a)\end{aligned} \tag{B.4}$$

where $b_{+1} = (D_z b)/a$.

References

[1] S.C. Anco, S. Seifi, A. Dar, New conserved integrals and invariants of radial compressible fluid flow and gas dynamics in $n > 1$ dimensions, (2022). arXiv: 2205.10950

[2] S.C. Anco, A. Dar, Classification of conservation laws of compressible isentropic fluid flow in $n > 1$ spatial dimensions, Proc. Roy. Soc. A 465 (2009), 2461–2488.

[3] S.C. Anco, A. Dar, Conservation laws of inviscid non-isentropic compressible fluid flow in $n > 1$ spatial dimensions, Proc. Roy. Soc. A 466 (2010), 2605–2632.

[4] S.C. Anco, A. Dar, N. Tufail, Conserved integrals for inviscid compressible fluid flow in Riemannian manifolds, Proc. Roy. Soc. A 471 (2015), 20150223 (24 pp).

[5] S.C. Anco, Generalization of Noether's theorem in modern form to non-variational partial differential equations. In: Recent progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science, 119–182, Fields Institute Communications, Volume 79 (2017).

[6] V.I. Arnold, The Hamiltonian nature of the Euler equation in the dynamics of rigid body and of an ideal fluid, Uspekhi Mat. Nauk 24 (1969), 225–226.

[7] V.I. Arnold, B.A. Khesin, *Topological Methods in Hydrodynamics*, Springer-Verlag, 1998.

[8] R.H. Aungier, *Turbine Aerodynamics: axial-flow and radial inflow turbine design and analysis*, ASME Press, 2005.

[9] G. Bluman, A. Cheviakov, S.C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Springer, 2009.

[10] B. Cantwell, *Introduction to Symmetry Analysis*, Cambridge University Press, 2002.

[11] M. Casey, C. Robinson, *Radial Flow Turbocompressors: Design, Analysis and Applications*, Cambridge University Press, 2021.

[12] A.A. Dezin, Invariant forms and some structure properties of the Euler equations of hydrodynamics, (in Russian) Zeit. Anal. Anwend. 2 (1983), 401–409.
[13] N.H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations Vol. 1,2,3*, CRC Press, 1994–1996.
[14] H.K. Jenssen, C. Tsikkou, On similarity flows for the compressible Euler system, J. Math. Phys. 59 (2018), 121507.
[15] J.B. Keller, Spherical, cylindrical and one-dimensional gas flows, Quart. Appl. Math. 14 (1956), 171–184.
[16] B.A. Khesin, Y.V. Chekanov, Invariants of the Euler equations for ideal or barotropic hydrodynamics and superconductivity in D dimensions, Physica D 40 (1989), 119–131.
[17] B.A. Kupershmidt, *The Variational Principles of Dynamics*, Advanced Series in Mathematical Physics vol. 13, World Scientific, 1992.
[18] Y. Kuroda, On the Casimir invariant of Hamiltonian fluid mechanics, J. Phys. Soc. Japan 60 (1991), 727–730.
[19] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, 1993.
[20] R.O. Popovych, V.M. Boyko, M.O. Nesterenko, M.W. Lutfullin, Realizations of real low-dimensional Lie algebras, J. Phys. A: Math. Gen. 36 (2003), 7337–7360.
[21] D. Serre, Invariants et dégénérescenc symplectique de l'équation d'Euler des fluids parfaits incompressibles, C.R. Acad. Sci. Paris, Sér. A 298 (1984), 349–352.
[22] G. Taylor, The dynamics of the combustion products behind plane and spherical detonation fronts in explosives, Proc. Roy. Soc. A 200 (1947), 235-247.
[23] J. Verosky, The Hamiltonian structure of generalized fluid equations, Lett. Math. Phys. 9 (1985), 51–53.
[24] R.L. Welsh, Imploding shocks and detonations, J. Fluid Mech. 29 (1967), 61–79.
[25] T. Wolf, in: Application of CRACK in the classification of integrable systems, 283–300, CRM Proceedings and Lecture Notes, Volume 37 (2004).
[26] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, 1974.