The Mathematics of Seki

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Outline

Definition and Examples of Seki

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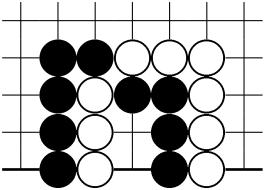
In its simple form, it is a sort of symbiosis where two live groups share liberties which neither of them can fill without dying.



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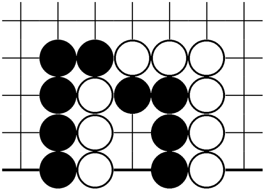

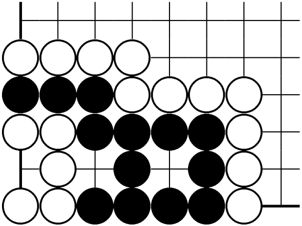

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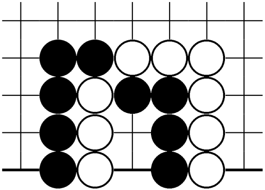

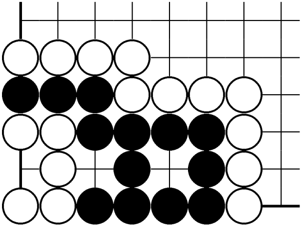

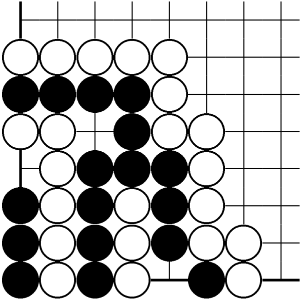

Hanezeki

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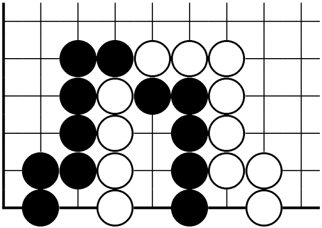
- non-terminal positions
- introduction of cuts

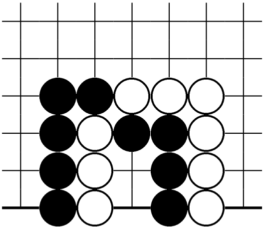
Outline

Non-terminal Positions

We are only interested in terminal positions.



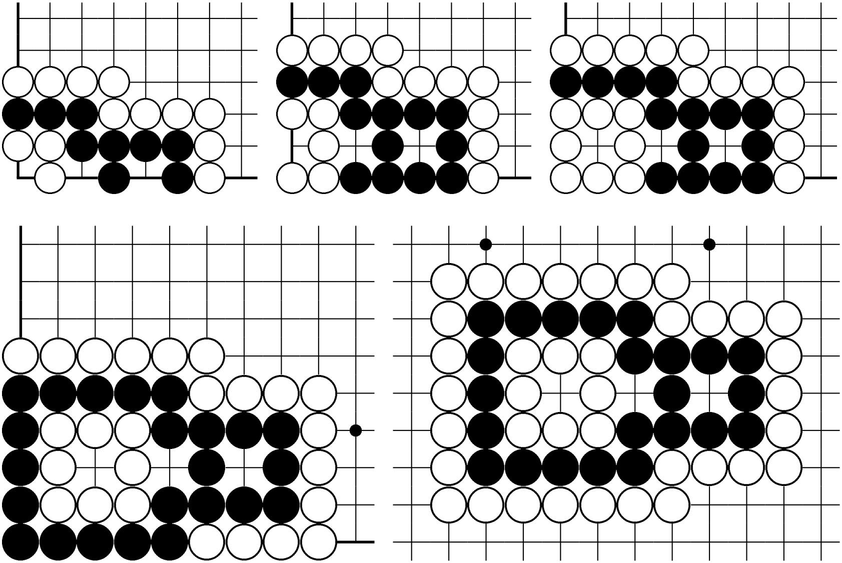
Non-terminal seki



Terminal seki

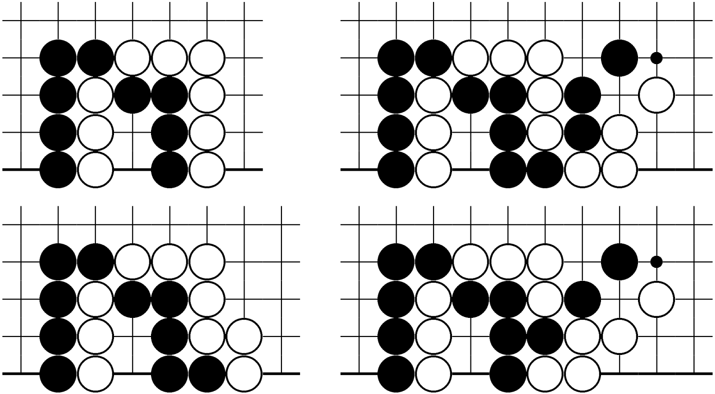
Shift and Deformation

All of these positions are equivalent.



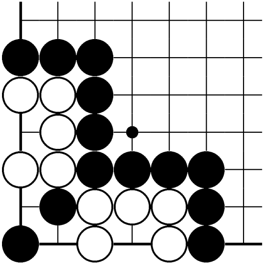
Introducing Cross Cuts I

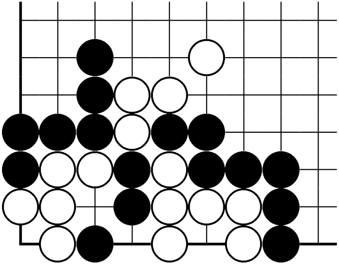
Also all of these seki are essentially identical despite two having a cross cut.

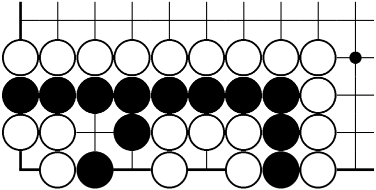


Introducing Cross Cuts II

The following positions differ even more but are still equivalent.







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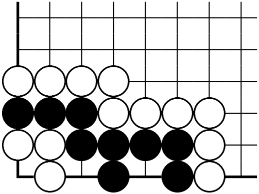
Commonly used in Go: the *Common Fate Graph* (CFG):



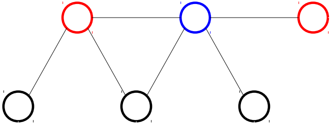


Figure: The corresponding CFG

Circles: red: white chain, blue: black chain, black: liberty
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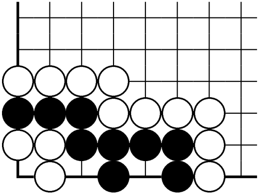

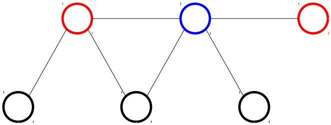

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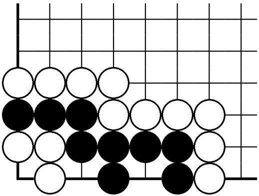

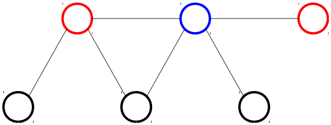

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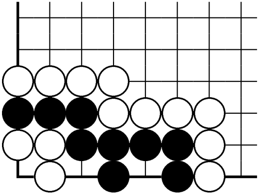

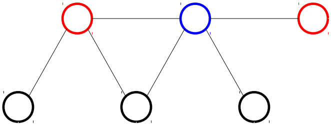

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But the choice of graph depends on the type of seki to be considered.

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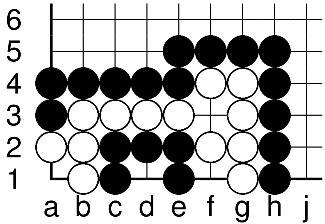
Positions are terminal, i.e. a move taking an opponent liberty gets instantly captured.

Basic Seki Graphs

This special class of seki allows more compact graphs: *Basic Seki Graphs* (BSG). Example:



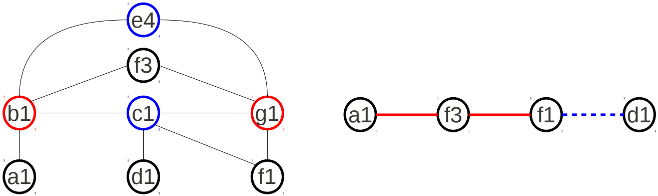


Figure: The 2 corresponding graphs: CFG and BSG

Properties of Basic Seki Graphs I

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- If two edges of same colour, say red, end in a shared node, say M, then both red edges must have their other end in the same other node, say N (otherwise White can move on M and give atari without being captured).

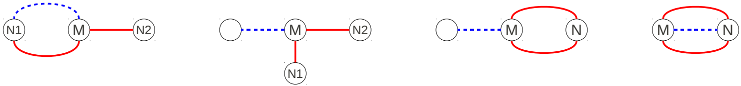

Figure: Two forbidden and two admissible graphs

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- If a node has edges of only one colour then these edges may reach only two other nodes (otherwise a move on *M* creates a chain with 3 liberties).

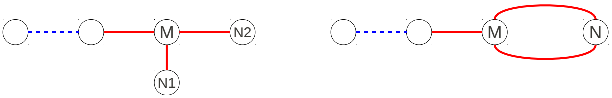

Figure: A forbidden and an admissible graph

Summary on Basic Seki

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- *Therefore Basic Seki consist either of a linear or a circular sequence of liberties where two neighbouring liberties are connected by only chains of one colour.*
- *The case of only 2 liberties connected by black and white chains can be seen as the smallest circular sequence.*

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A Numerical Encoding

Basic Seki are linear or circular (i.e. 1-dimensional)

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It turns out that conditions on Basic Seki Graphs shown before are not only necessary but also sufficient.

$\Rightarrow$ Generating all sequences of such number encodings will generate all Basic Seki.

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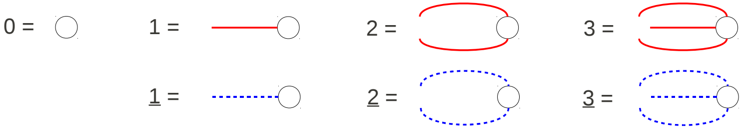

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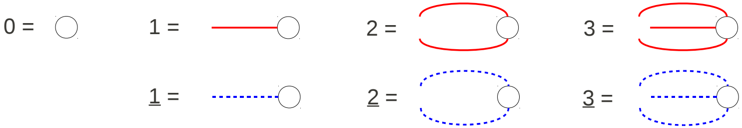

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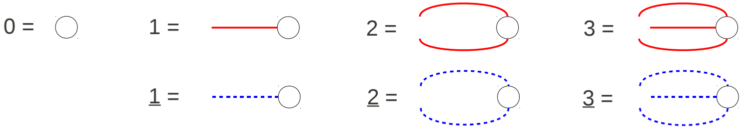

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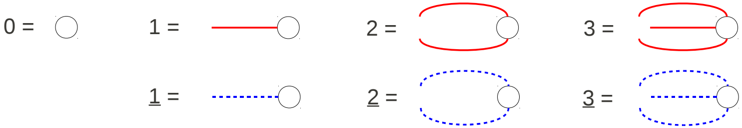

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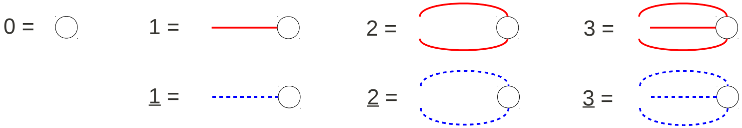

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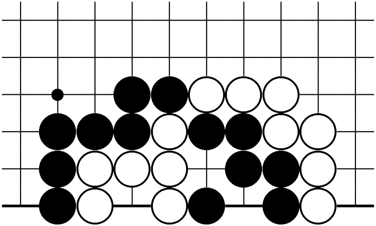
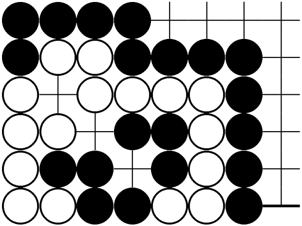
- two seki attached on board to one seki $\Rightarrow$ ... + ...

Examples of linear Seki I



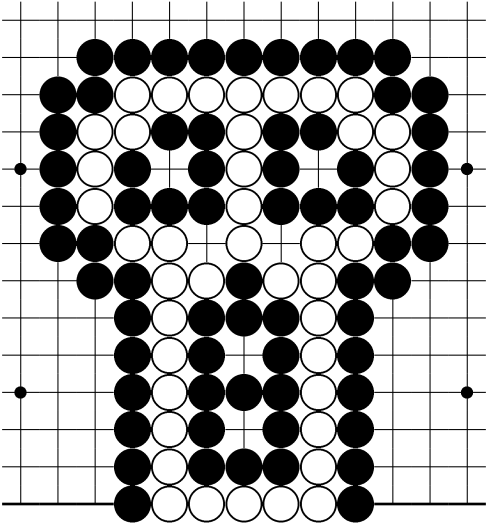
encoding: 01<u>2</u>

Examples of linear Seki II



encoding: 022

Examples of linear Seki III



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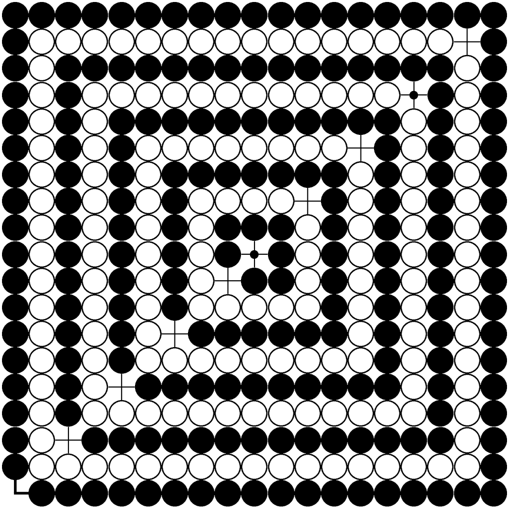
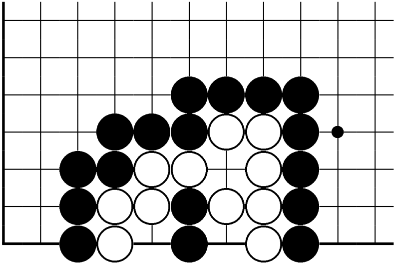
Examples of linear Seki IV



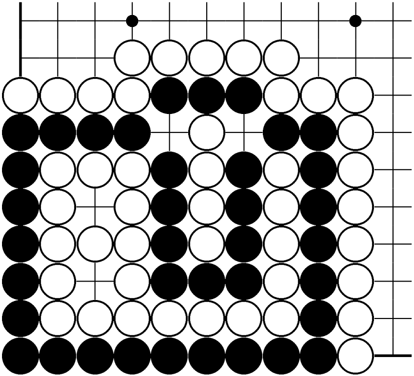
”The Onion” with encoding: $0\underline{1}1\underline{1}1\underline{1}1\underline{1}1\underline{1} = 0\underline{1}(1\underline{1})^4$
looks circular but is linear.

Examples of circular Seki I



encoding: 111

Examples of circular Seki II



encoding: 31

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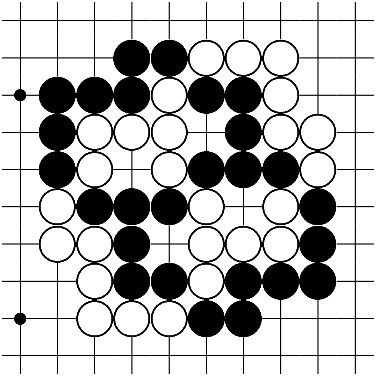

encoding: $\underline{1}1\underline{1}1 = (\underline{1}1)^2$

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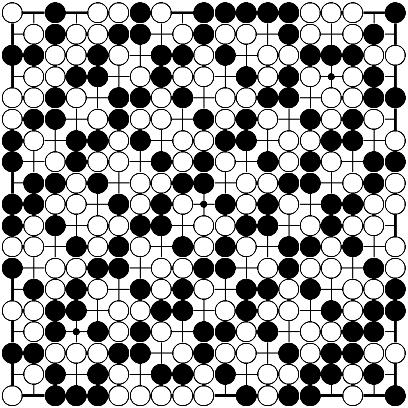
  - avoid identical circular basic seki (inversion, colour switch, cyclic permutation, e.g. $2\underline{1}1$ = $\underline{1}12$ = $12\underline{1}$ = ...)

Outline

Attaching Seki



A full board seki of G. Hungerink

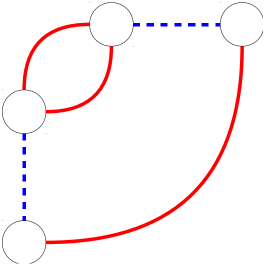
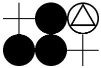


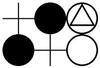
Figure: The upper left corner and the colour switched lower right corner of the board as BSG with encoding $\underline{1}2\underline{1}1$.

The BSG of the whole board consists of three disconnected sub graphs and has the encoding

$\underline{1}2\underline{1}1 + 0(\underline{2}2)^4\underline{1}(2\underline{2})^6 1\underline{1}2\underline{1}1(\underline{2}2)^6\underline{1}1\underline{2}1\underline{1}(2\underline{2})^6 1(\underline{2}2)^4 + \underline{1}1\underline{2}1.$

Cutting off a Stone I





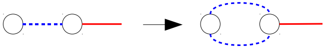
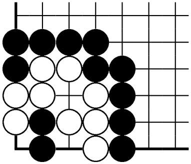
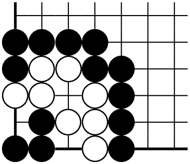


Figure: The change of BSG



encoding: 111



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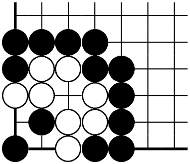
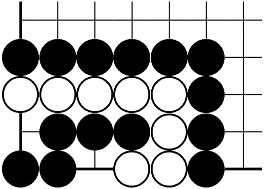
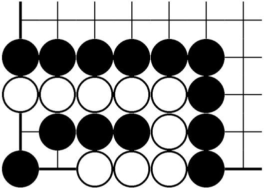


encoding: 211

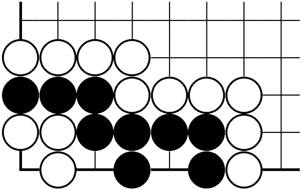
Cutting off a Stone II



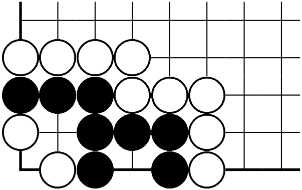
encoding: $\underline{1}1$



encoding: $\underline{2}1$



encoding: $01\underline{1}$



encoding: $02\underline{1}$

Creating an Eye I

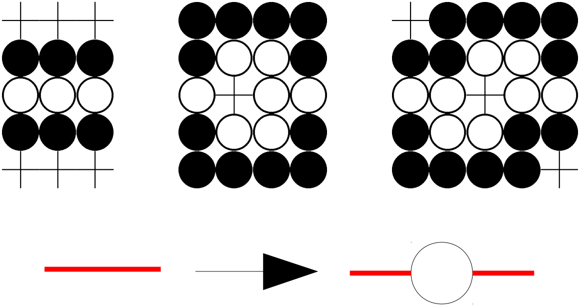
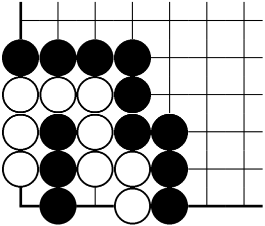
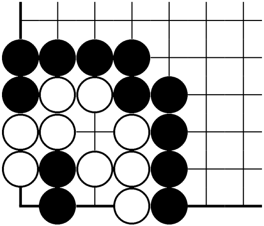


Figure: The change of CFG

Creating an Eye II



encoding: 11



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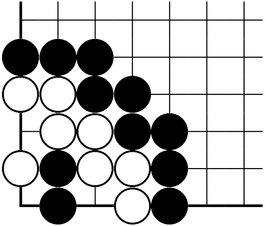

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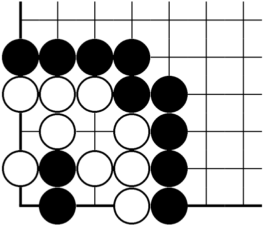
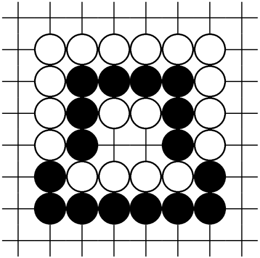
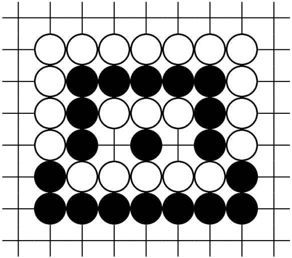


encoding: 1111

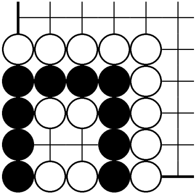
Bamboo Joints in Basic Seki I



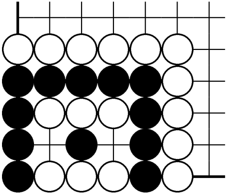
encoding: 21



encoding: 22

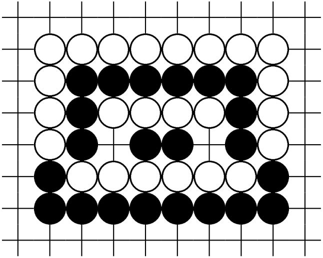


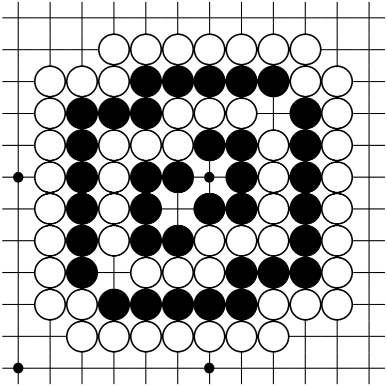
encoding 21



encoding: 22

Bamboo Joints in Basic Seki II





Both seki have the encoding $2\underline{2}$ but different sequences of black and white stones around liberties (WBWB and WWBB).

Outline

New Graphs needed I

So far: chains in Go $\rightarrow$ edges in graphs,
liberties in Go $\rightarrow$ nodes in graphs

(each chain had 2 liberties and each edge has 2 ends (nodes))

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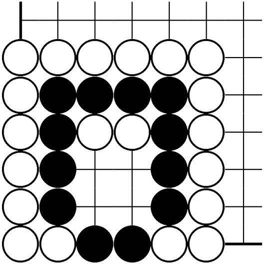

New Graphs needed II

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Before starting with *simple* graphs with only one edge between two nodes we give some theorems on regular graphs.

Outline

Planarity of Graphs

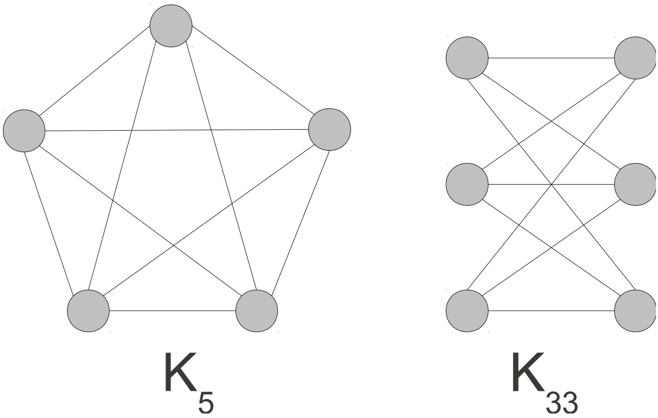

Figure: Forbidden sub-minors and sub-divisions of planar graphs

A graph is planar iff it does not contain a K_5 and no K_{33} sub division (sub minor).

Relationship to Terminal Seki

Theorem:
Each bi-partite 3-regular (planar) graph where each chain has at least two opponent neighbouring chains represents a terminal seki.

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Proof:
W.l.o.g. let us assume that White takes a joint liberty of chains △ and ▲. If now Black takes a liberty of △ from one of the other neighbours of △ then as a result, △ has only one liberty and all neighbours of △ have at least 2 liberties, i.e. White has no chance to safe △ .

Seki with a fixed Number of Liberties per Chain

Theorem:
In a position where each chain has the same number d of liberties the difference of the number of black and white eyes is equal d times the difference of numbers of black and white chains.

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Proof:
Let there be N_w white and N_b black chains, Y_w white and Y_b black (1-point) eyes, S shared liberties and let T_w, T_b be the total number of white and black liberties. Then $T_i = d \cdot N_i$ and $T_i = Y_i + S$. We therefore get $d \cdot N_w - Y_w = S = d \cdot N_b - Y_b$ and thus

$$Y_b - Y_w = d \cdot (N_b - N_w). \tag{1}$$

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$$Y_b - Y_w = d \cdot (N_b - N_w). \tag{1}$$

Corollary:
A position where each chain has the same number of liberties (> 1) can not have exactly one single eye.

Cuts with Flows I

Theorem:
Given a position where each liberty is either in a 1-point eye or is shared by exactly one white and one black chain and each chain has the same number of d liberties. Then any cut through the coresponding graph has a total flow determined through the number of chains and eyes on either side of the cut.

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Proof:

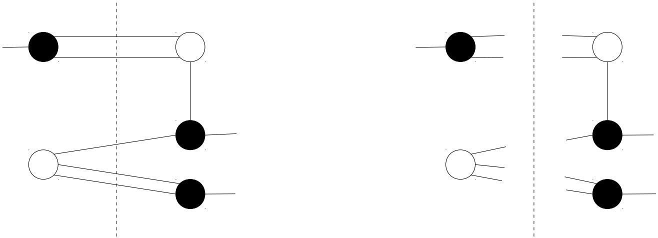


Figure: A cut of a 3-regular graph

Cuts with Flows II

Proof continued:

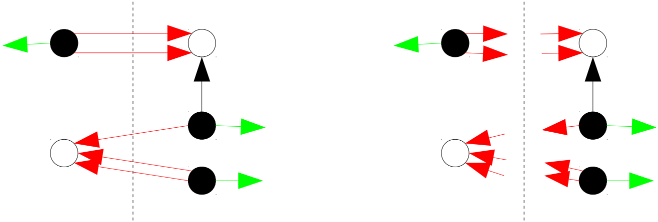
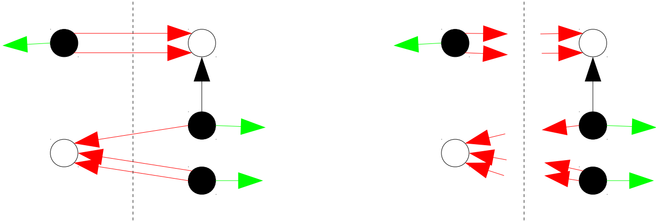


Figure: A cut of a 3-regular graph

On right side:
S_B^R ... number of liberties of Black in cutted edges
Y_B^R ... number of eyes of Black
N_B^R ... number of black chains
(similarly $_W$ for White and ^{L} for the left side)

Cuts with Flows III

Proof continued:



Replacing in the previous theorem the number of eyes Y_I^J by liberties in cutted edges plus number of eyes: $S_I^J + Y_I^J$ then we get for the total flow F through the cut defined by
$F := S_B^L - S_W^L = S_W^R - S_B^R$:

$$\begin{aligned} F &= (N_B^L - N_W^L)d - (Y_B^L - Y_W^L) = (1-1)3 - (1-0) = -1 \\ &= (N_W^R - N_B^R)d - (Y_W^R - Y_B^R) = (1-2)3 - (0-2) = -1 \end{aligned}$$

($\equiv$ discrete version of Gauß's Theorem)

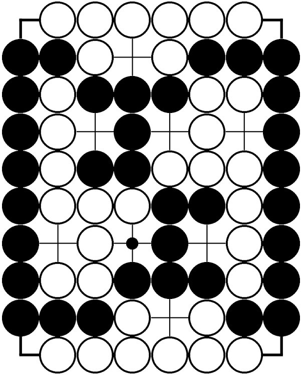
Outline

Seki with at most one shared Liberty between any two Chains

We start with *simple* graphs having only one edge between two nodes (i.e. seki where 2 chains share at most one liberty).

Bi-partite planar 3-regular Graphs

Sensei's Library [4]:



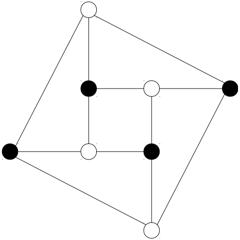


Figure: The corresponding Graph

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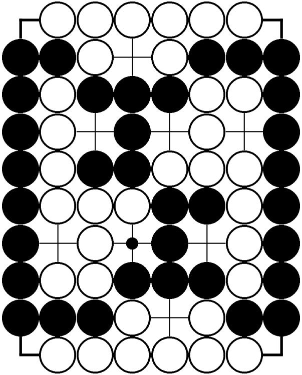

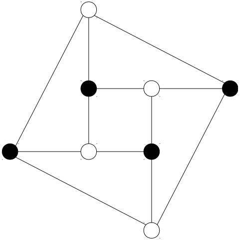

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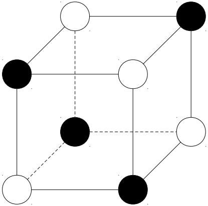
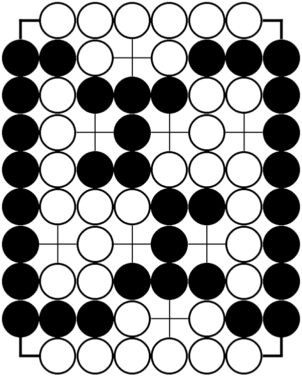
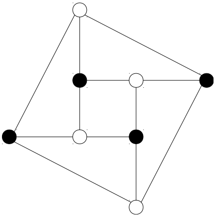


Figure: The same Graph

Planar Graphs and their Dual





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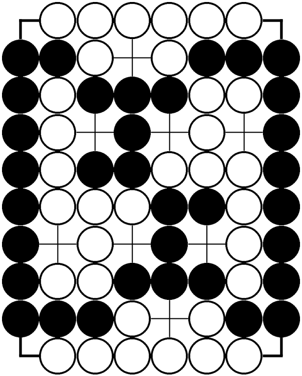

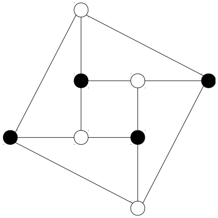

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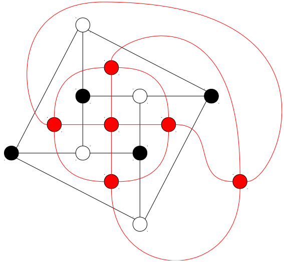

Figure: with it's dual Graph

The Cube and the Octahedron

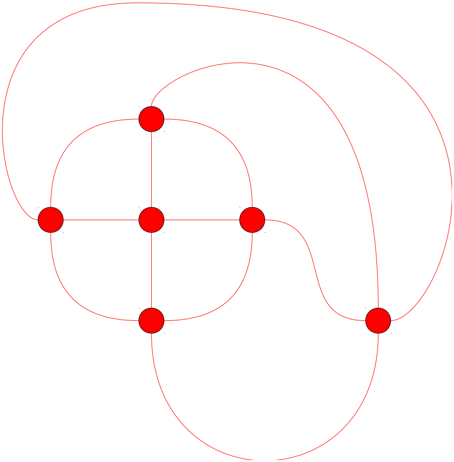

Figure: This dual Graph of a Cube

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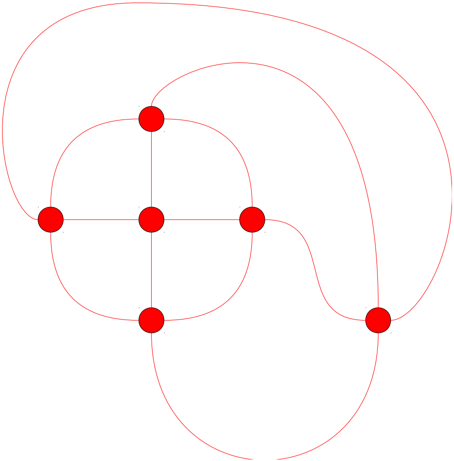

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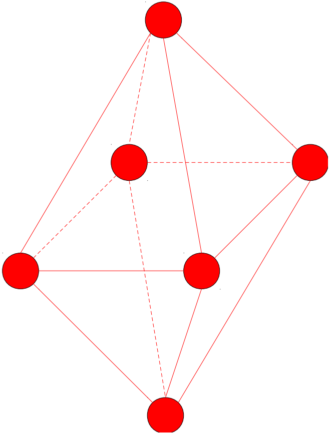


Figure: is an Octahedron

More simple bi-partite planar 3-regular Graphs I

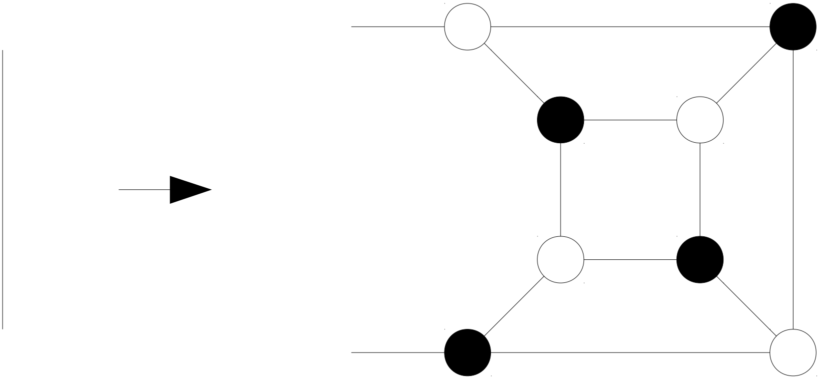
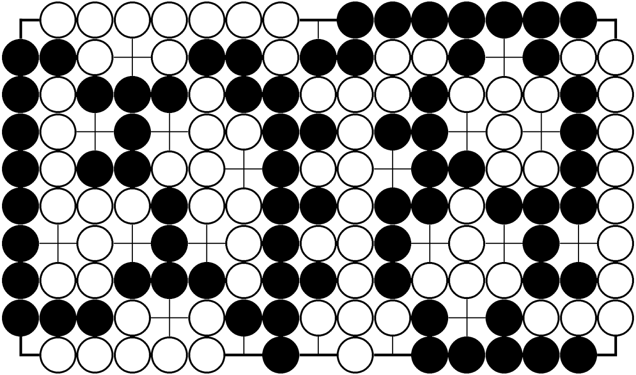


Figure: This replacement of any edge generates a new graph and thus a new seki. The right graph represents a seki (with 2 eyes) on its own.

More simple bi-partite planar 3-regular Graphs II



The position resulting from the complication step.

More simple bi-partite planar 3-regular Graphs III

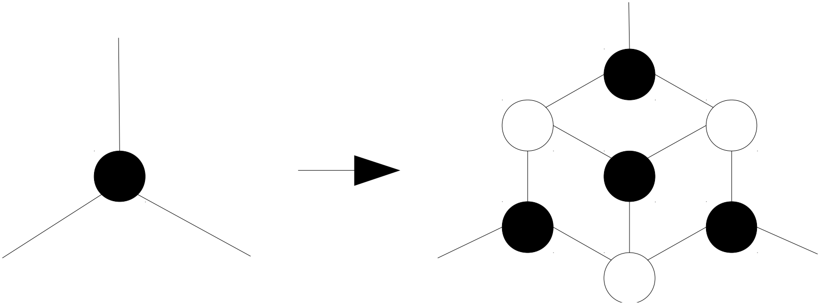


Figure: This replacement of any node also generates a new graph and thus a new seki. The right graph represents a seki (with 3 eyes) on its own.

Higher regular Graphs I

How about sekis corresponding to simple 4-regular bi-partite graphs where chains have each 4 liberties and any two chains share at most one liberty?

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Let $G = (V, E)$ be a connected planar graph with vertices V, edges E and set of regions R.

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Let $G = (V, E)$ be a connected planar graph with vertices V, edges E and set of regions R.
Euler's formula says: $|V| - |E| + |R| = 2$

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Proof:
Let $G = (V, E)$ be a connected planar graph with vertices V, edges E and set of regions R.
Euler's formula says: $|V| - |E| + |R| = 2$
Assuming G is simple and bipartite, each region is bounded by at least 4 edges.

$$4|R| \leq \sum_{r \in R} (\#\text{ of edges bounding region } r) \ = 2|E|$$

$$\Rightarrow 2|R| \leq |E|$$

Higher regular Graphs II

If G is at least 4-regular.

$$4|V| \leq \sum_{v \in V} \text{degree}(v) = 2|E|$$

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By Euler's formula:

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$\Rightarrow$ contradiction.

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$\Rightarrow$ contradiction.
In other words, there are no seki with chains having each the same number of 4 or more liberties, having no eyes and only 1 shared liberty between any 2 chains.

Outline

Higher regular but non-simple Graphs

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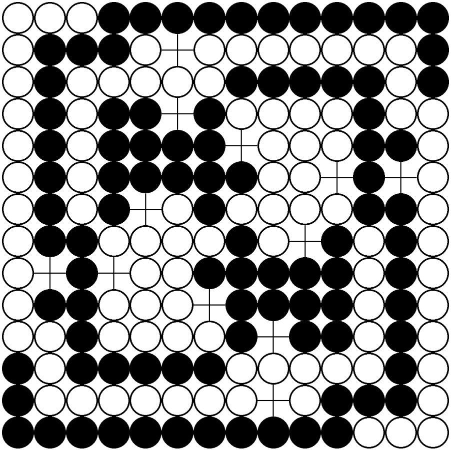
Bi-partite planar 3-regular graphs have a perfect matching.

$\Rightarrow$ opportunity to generate higher regular graphs with multi-edges. (i.e. seki with pairs of chains sharing more than one liberty).

Again a bi-partite planar 3-regular Graph



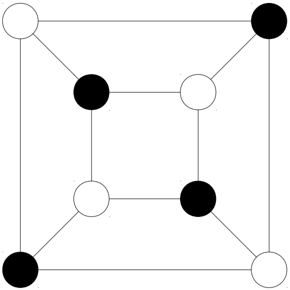
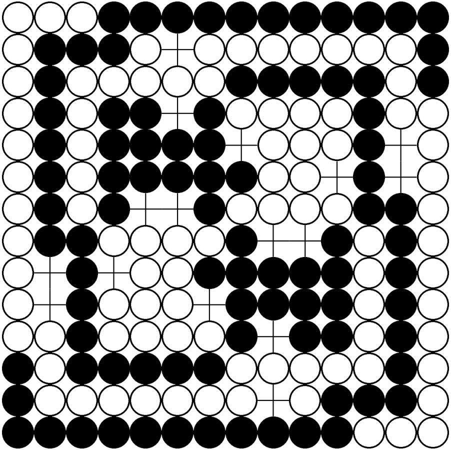


Figure: A cubical graph

A bi-partite planar 4-regular Graph



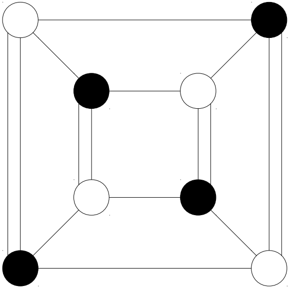
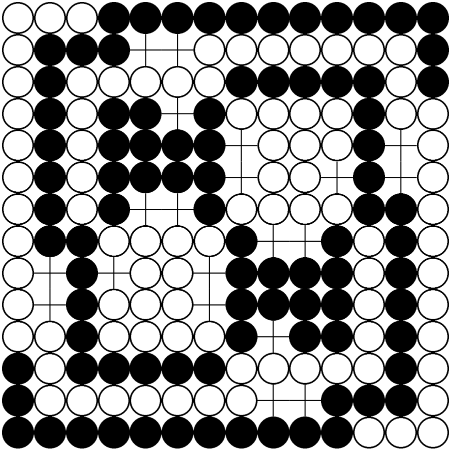


Figure: A cubical graph

A bi-partite planar 5-regular Graph



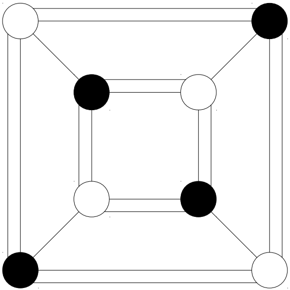
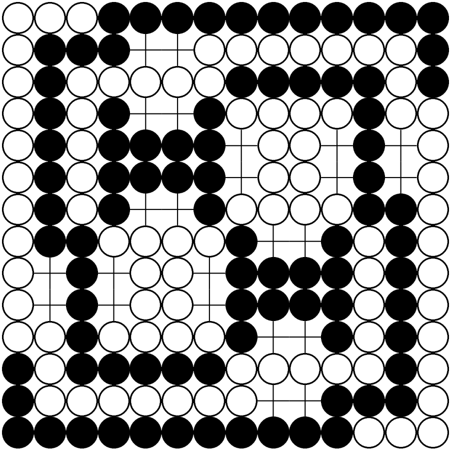


Figure: A cubical graph

A bi-partite planar 6-regular Graph



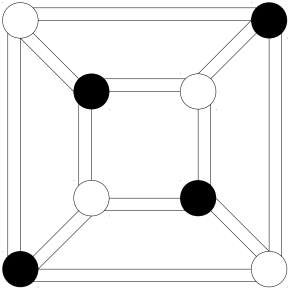
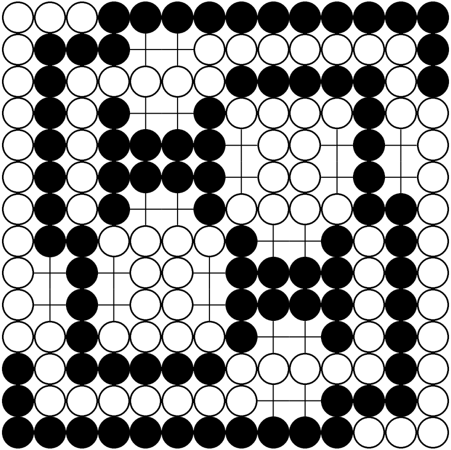


Figure: A cubical graph

A bi-partite planar 6-regular Graph



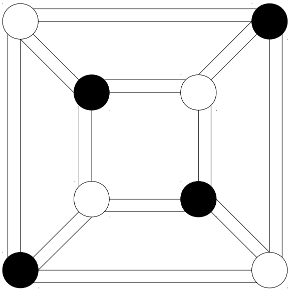


Figure: A cubical graph

Any matchings of the cubical graph involving 3 joint liberties between opponent chains give non-terminal seki.

More non-simple bi-partite planar 3-regular Graphs

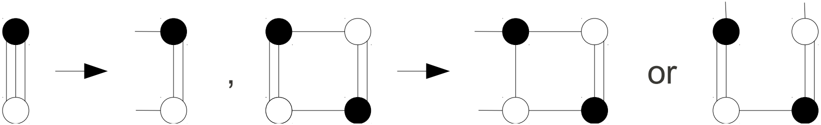


Figure: Take any 3-regular bi-partite graph like the two in this figure, cut any edge and use that to replace any edge in any other 3-regular bipartite graph to generate a new 3-regular bi-partite graph, i.e. a terminal seki. (If the lose ends are eyes then only the middle seki is terminal.)

Example: Benzol Variations

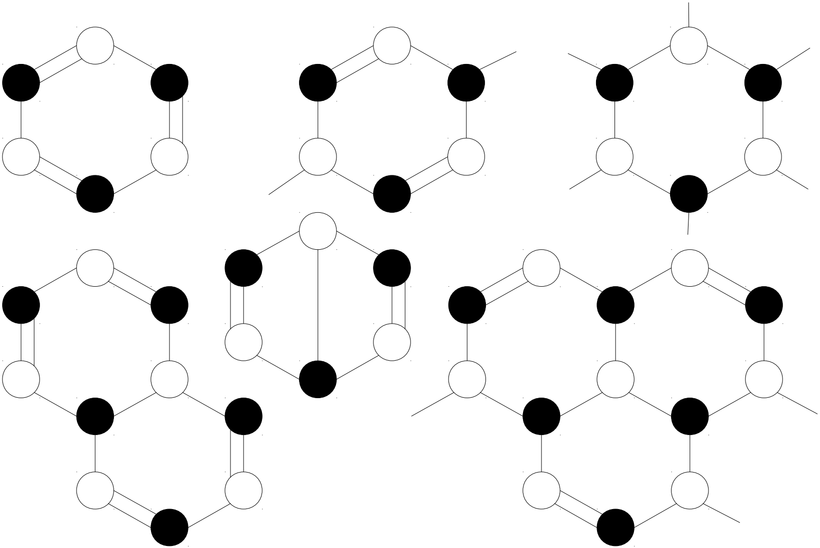


Figure: Terminal seki with a honey comb inside

Inserts in 3-regular Graphs

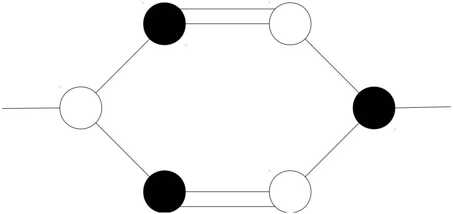


Figure: A multi-edge replacement for an edge in a 3-regular graph is a terminal seki on its own.

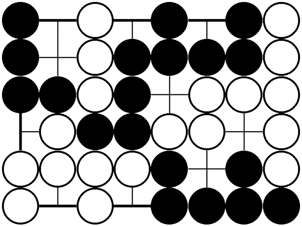


Diagram 1. A realization on a Go board with an eye on each side

3-regular Seki Creations

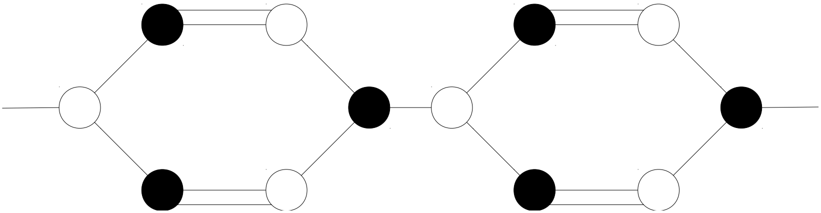


Figure: Such graphs of arbitrary length represent terminal seki.

4-regular Inserts

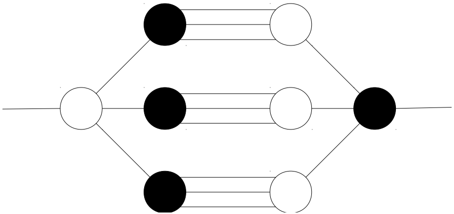


Figure: A terminal seki with two eyes, also when ends are connected

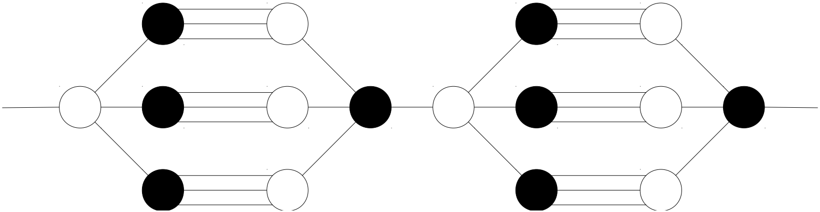


Figure: Any such creation is a terminal seki

A 5-regular Graph

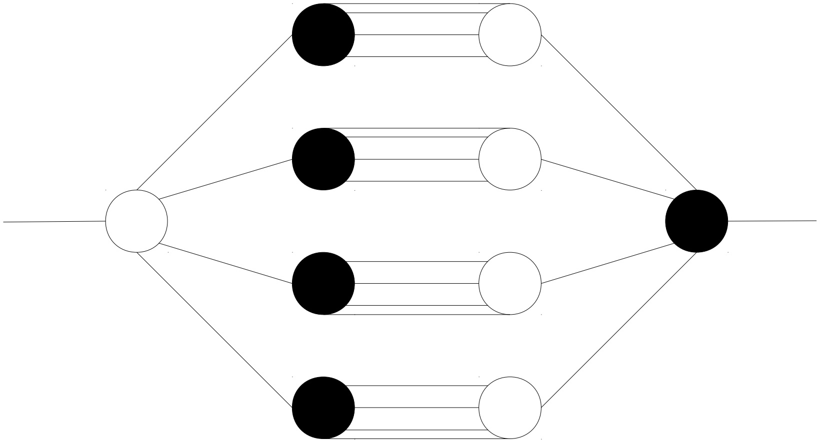


Figure: This is not a seki. Chains on the right and left can be captured.

Outline

Work of Vladimir Gurvich et. al. I

In preprint [8] (2012) and earlier preprints they consider positions

- without eyes
- where each liberty has exactly one white and one black neighbouring chain.

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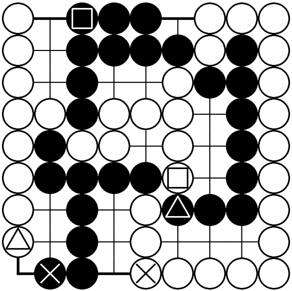
Moves are made by decreasing an A_{ij} by 1.

Problem: For a given computer determined seki matrix a Go position may not exist, e.g. not for:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Work of Vladimir Gurvich et. al. II

Example:



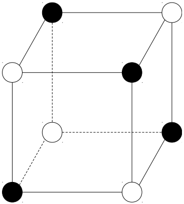
$_i\backslash^j$	△	⊗	□	s_i^B
▲	0	3	3	6
✖	3	3	1	7
■	3	1	2	6
s_j^W	6	7	6	

Table: The liberty matrix

Lemma: (giving sufficient conditions for Black to capture)
Even when playing second, Black captures if there is a column j such that $s_i^B - A_{ij} \geq s_j^W$ for every row i and $s_i^B > s_j^W$ if $A_{ij} = 0$.

Outline

‘Local Seki’ versus ‘global Seki’



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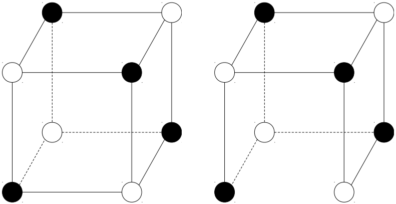

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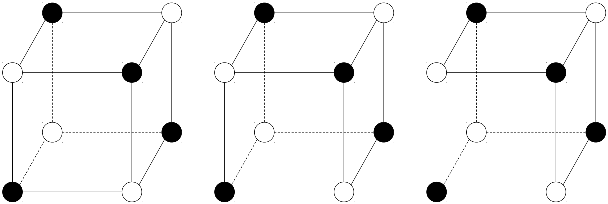

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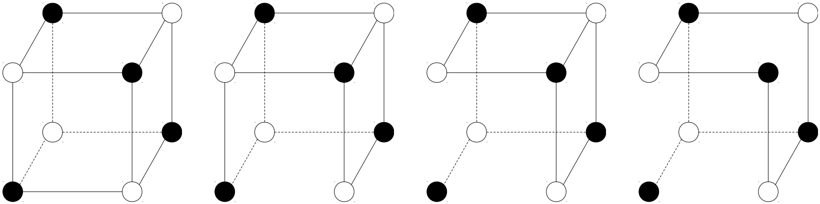

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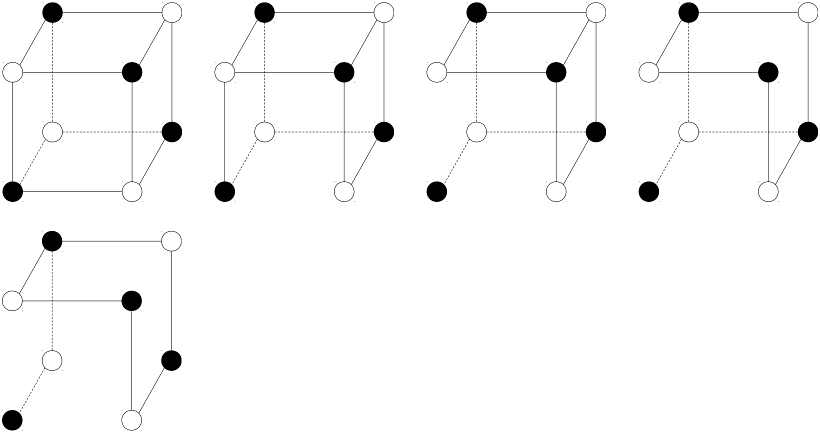

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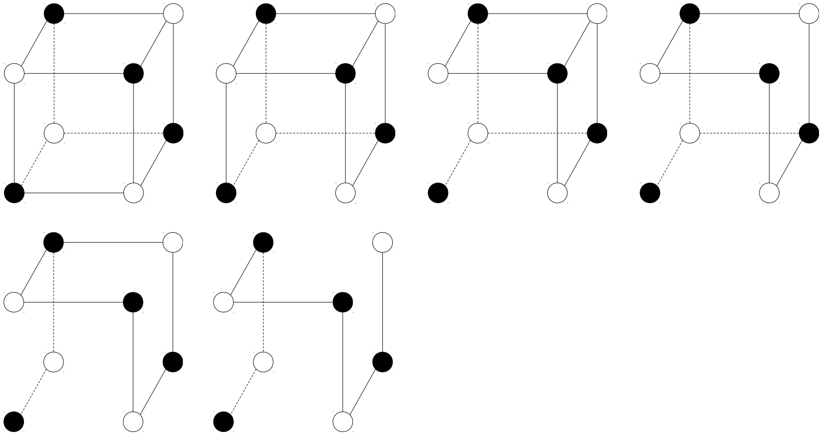

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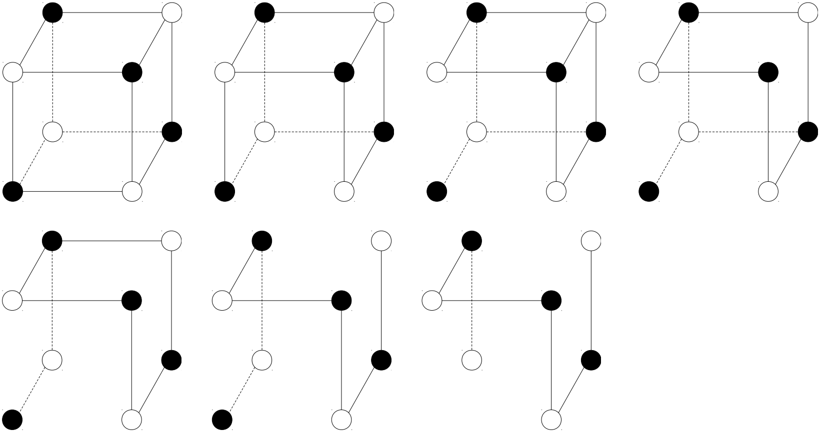

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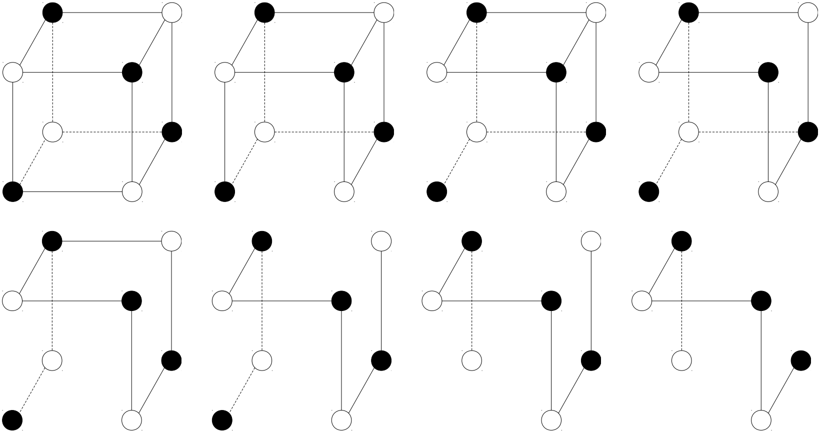


Figure: Global instability of Cubical Seki

$\Rightarrow$ Each attacking chain is captured (i.e. is a local seki) but in return an opponent chain can be captured (i.e. no global seki).

‘Local Seki’ versus ‘global Seki’

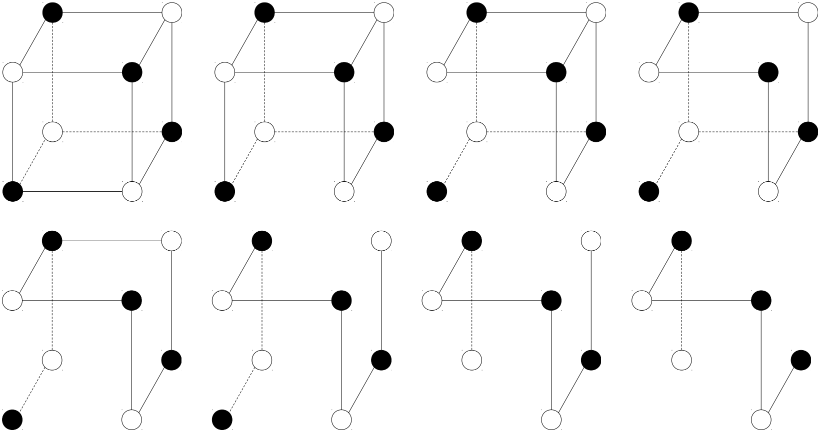
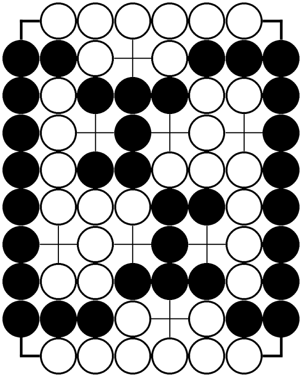


Figure: Global instability of Cubical Seki

⇒ Each attacking chain is captured (i.e. is a local seki) but in return an opponent chain can be captured (i.e. no global seki).

⇒ Sacrifice a small chain and catch a big one ⇒ no “real” seki.

A Sacrifice in a Local Seki



Black to play and sacrifice a small chain to catch a bigger one.

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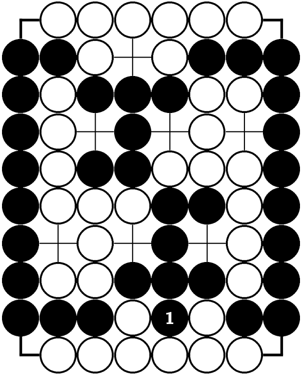

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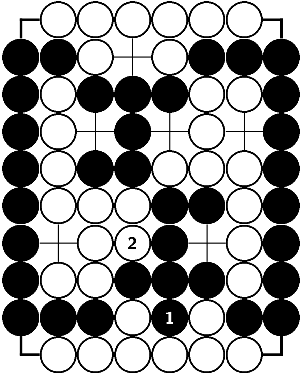

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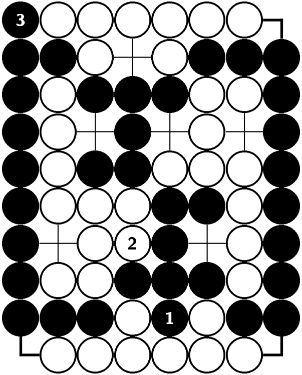

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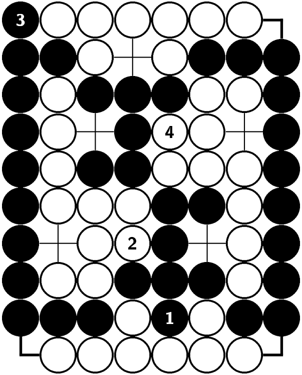

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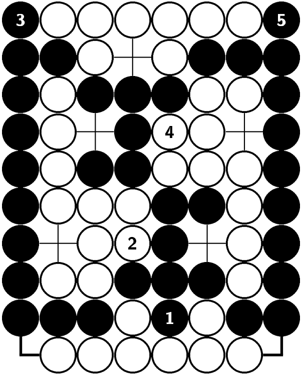

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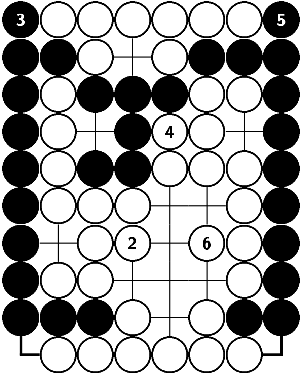

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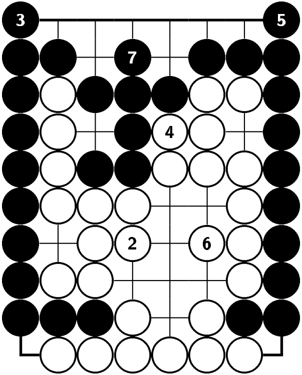

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Outline

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The End

Thank you!