

Journal of Nonlinear Mathematical Physics, Vol. , No. (2012) 1–12  
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## THE TWO-COMPONENT CAMASSA-HOLM EQUATIONS CH(2,1) AND CH(2,2): FIRST-ORDER INTEGRATING FACTORS AND CONSERVATION LAWS

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Received (8 March 2012)  
 Revised (2 September 2012)  
 Accepted (Day Month Year)

Recently, Holm and Ivanov, proposed and studied a class of multi-component generalisations of the Camassa-Holm equations [D D Holm and R I Ivanov, Multi-component generalizations of the CH equation: geometrical aspects, peakons and numerical examples, *J. Phys A: Math. Theor* **43**, 492001 (20pp), 2010]. We consider two of those systems, denoted by Holm and Ivanov by CH(2,1) and CH(2,2), and report a class of integrating factors and its corresponding conservation laws for these two systems. In particular, we obtain the complete set of first-order integrating factors for the systems in Cauchy-Kovalevskaya form and evaluate the corresponding sets of conservation laws for CH(2,1) and CH(2,2).

*Keywords:* Multi-component Camassa-Holm equations; Conservation laws; Integrating factors

2000 Mathematics Subject Classification: 35Q35, 76B15

### 1. Introduction

It is well known that certain conservation laws of shallow water wave equations, such as the Camassa-Holm equation [4] and the the Degasperis-Procesi equation [8], are useful to prove blow-up, cf. the papers [5], [18] and [15]. Furthermore, conservation laws play a central role in the prove of the global existence (in time) for solutions evolving from certain initial data, cf. the paper [6], and for proving the stability of peakons for both model equations, cf. the papers [7], [12] and [13]. In the context of the Camassa-Holm equation they are instrumental in the set-up of a theory of global weak solutions for nonlinear nonlocal conservation laws, cf. the considerations in the papers [2], [3] and [10]

In the current paper we derive all first-order integrating factors and its corresponding conservation laws for some recently proposed multi-component generalizations of the

Camassa-Holm equation [11]. We concentrate on two explicit systems, namely CH(2,1) and CH(2,2), proposed by Holm and Ivanov in [11] (see 1.1a) – (1.1b) and (1.6a) – (1.6b) below).

We recently reported in [9] the complete set of first-order integrating factors and conservation laws for a class of Camassa-Holm type equations, which includes the Camassa-Holm equation [4] and the the Degasperis-Procesi equation [8]. Our approach applied in this paper is based on the direct method described by Anco and Bluman in their paper [1], which can be applied to derive conservation laws of evolution equations that are in Cauchy-Kovalevskaya form. We also refer the reader to [16] and [17] for more details and alternate methods for computing conservation laws for partial differential equations and systems.

Consider the two-component Camassa-Holm equations introduced and denoted by Holm and Ivanov [11] as CH(2,1), which has the following form:

$$\sigma_1 q_t + 2qu_x + uq_x + \sigma\rho\rho_x = 0 \quad (1.1a)$$

$$\rho_t + \rho u_x + u\rho_x = 0, \quad (1.1b)$$

where

$$q = \sigma_1 u - u_{xx} + s \quad (1.2)$$

and  $s$ ,  $\sigma$  and  $\sigma_1$  are arbitrary constants. The physically interesting cases are  $\sigma = \pm 1$  and  $\sigma_1 = 1$  or  $\sigma_1 = 0$ . By defining the new dependent variables

$$u := U_1, \quad u_x := U_2 \quad (1.3a)$$

$$u_{xx} := U_3, \quad \rho := U_4 \quad (1.3b)$$

and the change of independent variables,

$$X := t, \quad T := x, \quad (1.4)$$

we can write system (1.1a) – (1.1b) in the following Cauchy-Kovalevskaya form:

$$E_1 := U_{1,T} - U_2 = 0 \quad (1.5a)$$

$$E_2 := U_{2,T} - U_3 = 0 \quad (1.5b)$$

$$\begin{aligned} E_3 := & U_{3,T} - \sigma_1^2 U_1^{-1} U_{1,X} + \sigma_1 U_1^{-1} U_{3,X} - 3\sigma_1 U_2 + 2U_1^{-1} U_2 U_3 + \sigma U_1^{-2} U_4 U_{4,X} \\ & + \sigma U_1^{-2} U_2 U_4^2 - 2s U_1^{-1} U_2 = 0 \end{aligned} \quad (1.5c)$$

$$E_4 := U_{4,T} + U_1^{-1} U_{4,X} + U_1^{-1} U_2 U_4 = 0. \quad (1.5d)$$

The second 2-component Camassa-Holm equation that we study in the current paper, denoted by CH(2,2), has the form [11]

$$q_{1,t} + u_0 q_{1,x} + 2q_1 u_{0,x} + u_1 q_{2,x} + 2q_2 u_{1,x} = 0 \quad (1.6a)$$

$$q_{2,t} + u_0 q_{2,x} + 2q_2 u_{0,x} = 0, \quad (1.6b)$$

where

$$q_1 = u_1 - u_{1,xx} + s_1 \quad (1.7a)$$

$$q_2 = u_0 - u_{0,xx} + 3u_1^2 - u_{1x}^2 - 2u_1u_{1,xx} + 4s_1u_1 + s_2. \quad (1.7b)$$

Here  $s_1, s_2$  are arbitrary constants. By defining the new dependent variables

$$u_0 := U_1, \quad u_{0,x} := U_2, \quad u_{0,xx} := U_3 \quad (1.8a)$$

$$u_1 := U_4, \quad u_{1,x} := U_5, \quad u_{1,xx} := U_6 \quad (1.8b)$$

and the change of independent variables (1.4), we can present (1.6a) – (1.6b) in the following Cauchy-Kovalevskaya form:

$$E_1 := U_{1,T} - U_2 = 0 \quad (1.9a)$$

$$E_2 := U_{2,T} - U_3 = 0 \quad (1.9b)$$

$$\begin{aligned} E_3 := & U_{3,T} + 12U_1^{-1}U_4^3U_5 - 4U_1^{-1}U_4U_{4,X} + 2U_1^{-1}U_5U_{5,X} - 4s_1U_1^{-1}U_{4,X} \\ & + 4U_5U_6 - 4s_1U_5 + 2U_1^{-1}U_2U_3 - 6U_1^{-1}U_2U_4^2 + 2U_1^{-1}U_2U_5^2 - 2s_2U_1^{-1}U_2 \\ & - 4s_1U_1^{-1}U_2U_4 - 12U_1^{-2}U_2U_4^4 + 2U_1^{-1}U_6U_{4,X} - 8U_1^{-1}U_4^2U_5U_6 \\ & + 16s_1U_1^{-1}U_4^2U_5 + 4U_1^{-2}U_4^2U_6U_{4,X} - 8s_1U_1^{-2}U_4^2U_{4,X} + 4U_1^{-2}U_4^2U_2U_3 \\ & + 4U_1^{-2}U_4^2U_2U_5^2 + 8U_1^{-2}U_2U_4^3U_6 - 16s_1U_1^{-2}U_2U_4^3 - 4s_2U_1^{-2}U_2U_4^2 \\ & + 4U_1^{-2}U_4^2U_5U_{5,X} - 4U_1^{-1}U_3U_4U_5 + 4s_2U_1^{-1}U_4U_5 - 12U_1^{-2}U_4^3U_{4,X} \\ & + 2U_1^{-2}U_4^2U_{3,X} + 4U_1^{-2}U_4^3U_{6,X} - 4U_1^{-1}U_4U_5^3 - 2U_1^{-2}U_4^2U_{1,X} \\ & - U_1^{-1}U_{1,X} + U_1^{-1}U_{3,X} - 3U_2 = 0 \end{aligned} \quad (1.9c)$$

$$E_4 := U_{4,T} - U_5 = 0 \quad (1.9d)$$

$$E_5 := U_{5,T} - U_6 = 0 \quad (1.9e)$$

$$\begin{aligned} E_6 := & U_{6,T} + 4U_1^{-1}U_4U_5U_6 - 8s_1U_1^{-1}U_4U_5 + 2U_1^{-1}U_5^3 - 3U_5 - U_1^{-1}U_{4,X} \\ & + U_1^{-1}U_{6,X} - 2U_1^{-2}U_4^2U_{6,X} + 6U_1^{-2}U_2U_4^3 - U_1^{-2}U_4U_{3,X} + U_1^{-2}U_4U_{1,X} \\ & + 6U_1^{-2}U_4^2U_{4,X} - 2U_1^{-2}U_4U_6U_{4,X} + 4s_1U_1^{-2}U_4U_{4,X} - 2U_1^{-2}U_2U_3U_4 \\ & - 2U_1^{-2}U_2U_4U_5^2 - 4U_1^{-2}U_2U_4^2U_6 + 8s_1U_1^{-2}U_2U_4^2 + 2s_2U_1^{-2}U_2U_4 \\ & - 2U_1^{-2}U_4U_5U_{5,X} + 2U_1^{-1}U_3U_5 - 2s_2U_1^{-1}U_5 + 2U_1^{-1}U_2U_6 \\ & - 2s_1U_1^{-1}U_2 - 6U_1^{-1}U_4^2U_5 = 0. \end{aligned} \quad (1.9f)$$

The above first-order Cauchy-Kovalevskaya systems can now be investigated for integrating factors to derive conservation laws for the systems; which then leads to conservation laws of the systems CH(1,1) and CH(2,2) in the original variables.

## 2. General description

In this section we briefly describe the direct method [1] of integrating factors (or multipliers) for the general first-order Cauchy-Kovalevskaya system of six equations:

$$E_j := U_{j,T} - F_j(U_1, \dots, U_6, U_{1,X}, \dots, U_{6,X}) = 0, \quad j = 1, 2, \dots, 6. \quad (2.1)$$

Every conserved density,  $\Phi^T$ , and conserved flux,  $\Phi^X$ , of system (2.1) must satisfy

$$D_T \Phi^T + D_X \Phi^X \Big|_{\vec{E}=\vec{0}} = 0, \quad (2.2)$$

where, in general, both  $\Phi^T$  and  $\Phi^X$  are functions of  $X, T, U_j$  as well as  $X$ -derivatives of  $U_j$ . Moreover, every  $\Phi^T$  requires six integrating factors,  $\{\Lambda_1, \Lambda_2, \dots, \Lambda_6\}$ , which are directly related to the conserved density by the relation [1]

$$\Lambda_k = \hat{E}[U_k] \Phi^T, \quad k = 1, 2, \dots, 6. \quad (2.3)$$

Here  $\hat{E}$  is the Euler Operator,

$$\hat{E}[U_k] := \frac{\partial}{\partial U_k} - D_T \circ \frac{\partial}{\partial U_{k,T}} + \sum_{j=1}^q (-1)^j D_X^j \circ \frac{\partial}{\partial U_{k,jX}}, \quad (2.4)$$

where we use the notation

$$U_{k,jX} := \frac{\partial^j U_k}{\partial X^j}.$$

The conditions on the integrating factors,  $\{\Lambda_j\}$ , of system (2.1) are

$$\hat{E}[U_k] (\Lambda_1 E_1 + \Lambda_2 E_2 + \dots + \Lambda_6 E_6) = 0, \quad k = 1, 2, \dots, 6. \quad (2.5)$$

However, since all integrating factors of system (2.1) are adjoint symmetries of the system (2.1), we can calculate  $\{\Lambda_j\}$  by the condition

$$\begin{pmatrix} L_{E_1}^*[U_1] & L_{E_2}^*[U_1] & \dots & L_{E_6}^*[U_1] \\ L_{E_1}^*[U_2] & L_{E_2}^*[U_2] & \dots & L_{E_6}^*[U_2] \\ \vdots & \vdots & \vdots & \vdots \\ L_{E_1}^*[U_6] & L_{E_2}^*[U_6] & \dots & L_{E_6}^*[U_6] \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_6 \end{pmatrix} \Big|_{\vec{E}=\vec{0}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (2.6)$$

and then require the self-adjointness condition on  $\{\Lambda_j\}$  (as integrating factors are variational quantities), namely

$$\begin{aligned} & \begin{pmatrix} L_{\Lambda_1}[U_1] & L_{\Lambda_1}[U_2] & \cdots & L_{\Lambda_1}[U_6] \\ L_{\Lambda_2}[U_1] & L_{\Lambda_2}[U_2] & \cdots & L_{\Lambda_2}[U_6] \\ \vdots & \vdots & \vdots & \vdots \\ L_{\Lambda_6}[U_1] & L_{\Lambda_6}[U_2] & \cdots & L_{\Lambda_6}[U_6] \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_6 \end{pmatrix} \\ &= \begin{pmatrix} L_{\Lambda_1}^*[U_1] & L_{\Lambda_2}^*[U_1] & \cdots & L_{\Lambda_6}^*[U_1] \\ L_{\Lambda_1}^*[U_2] & L_{\Lambda_2}^*[U_2] & \cdots & L_{\Lambda_6}^*[U_2] \\ \vdots & \vdots & \vdots & \vdots \\ L_{\Lambda_1}^*[U_6] & L_{\Lambda_2}^*[U_6] & \cdots & L_{\Lambda_6}^*[U_6] \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_6 \end{pmatrix}. \end{aligned} \quad (2.7)$$

$$(2.8)$$

Here  $L$  is the linear operator and  $L^*$  its adjoint:

$$L_P[U_j] := \frac{\partial P}{\partial U_j} + \sum_{i=1}^p \frac{\partial P}{\partial U_{j,iT}} D_T^i + \sum_{k=1}^q \frac{\partial P}{\partial U_{j,kX}} D_X^k \quad (2.9a)$$

$$L_P^*[U_j] := \frac{\partial P}{\partial U_j} + \sum_{i=1}^p (-1)^i D_T^i \circ \frac{\partial P}{\partial U_{j,iT}} + \sum_{k=1}^q (-1)^k D_X^k \circ \frac{\partial P}{\partial U_{j,kX}}. \quad (2.9b)$$

Note that the self-adjointness condition, (2.7), is independent of the form of the evolution system (2.1) and only depends on the functional arguments of  $\{\Lambda_j\}$  as well as the number of equations in the system.

### 3. Integrating factors for system (1.5a) – (1.5d) and conservation laws for (1.1a) – (1.1b):

Solving conditions (2.6) and (2.7) for system (1.5a) – (1.5d), the complete set of first-order integrating factors  $\{\Lambda_1, \dots, \Lambda_4\}$ , of the form

$$\Lambda_j = \Lambda_j(X, T, U_1, \dots, U_4, U_{1,X}, \dots, U_{4,X}), \quad j = 1, 2, \dots, 4$$

for arbitrary  $\sigma$ ,  $\sigma_1 \neq 0$  and  $s$  is as follows:

$$\begin{aligned} \Lambda_1 &= \lambda_1 U_4 + 2\lambda_2 \left( s + \frac{3}{2}\sigma_1 U_1 - \frac{1}{2}U_3 \right) \\ &- \lambda_3 (\sigma_1 U_{2,X} - 2sU_1 - \sigma U_4^2 - 3\sigma_1 U_1^2 + 2U_1 U_3) \sigma_1^{-1} \end{aligned} \quad (3.1a)$$

$$\Lambda_2 = -\lambda_2 U_2 + \lambda_3 U_{1,X} \quad (3.1b)$$

$$\Lambda_3 = -\lambda_2 U_1 - \lambda_3 \sigma_1^{-1} U_1^2 \quad (3.1c)$$

$$\Lambda_4 = \lambda_1 U_1 + \lambda_2 \sigma U_4 + 2\lambda_3 \sigma \sigma_1^{-1} U_1 U_4 \quad (3.1d)$$

where  $\lambda_j$  are arbitrary constants. This leads to the following three sets of conserved density,  $\Phi^t$ , and conserved flux,  $\Phi^x$ , for the original system (1.1a) – (1.1b) (separated by means of the arbitrary  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$ , respectively):

$$\Phi_1^t = \rho \quad (3.2a)$$

$$\Phi_1^x = u\rho \quad (3.2b)$$

$$\Phi_2^t = \sigma_1^2 u - \sigma_1 u_{xx} \quad (3.3a)$$

$$\Phi_2^x = 2su + \frac{1}{2}\sigma\rho^2 + \frac{3}{2}\sigma_1 u^2 - uu_{xx} - \frac{1}{2}u_x^2 \quad (3.3b)$$

$$\Phi_3^t = \left( -\sigma_1 u_x^2 + \frac{1}{2}\sigma\rho^2 + \frac{1}{2}\sigma_1^2 u^2 - \sigma_1 uu_{xx} + \frac{1}{2}\sigma_1 u_x^2 \right) \sigma_1^{-1} \quad (3.4a)$$

$$\Phi_3^x = (\sigma_1 u_x u_t + s u_x^2 + \sigma u \rho^2 + \sigma_1 u^3 - u^2 u_{xx}) \sigma_1^{-1} \quad (3.4b)$$

Some special must be considered:

**Special Case 1:**  $\sigma = 0$  with  $\sigma_1$  arbitrary, but nonzero, and  $s$  arbitrary. The integrating factors are as follows:

$$\Lambda_1 = \frac{2U_3 - 3\sigma_1 U_1 - 2s}{(\sigma_1 U_1 - U_3 + s)^{1/2}}, \quad \Lambda_2 = 0 \quad (3.5)$$

$$\Lambda_3 = \frac{U_1}{(\sigma_1 U_1 - U_3 + s)^{1/2}}, \quad \Lambda_4 = 0. \quad (3.6)$$

and the corresponding conserved current for system (1.5a) – (1.5d) is

$$\Phi^t = \sigma_1 (\sigma_1 u - u_{xx} + s)^{1/2} \quad (3.7a)$$

$$\Phi^x = (\sigma_1 u - u_{xx} + s)^{1/2}. \quad (3.7b)$$

**Special Case 2:**  $\sigma = 0$  with  $\sigma_1 = 1$  and  $s$  arbitrary. The integrating factors are as follows:

$$\Lambda_1 = -\frac{U_1 W^3 H'(W)}{U_4} + 2U_4 H(W), \quad \Lambda_2 = 0 \quad (3.8a)$$

$$\Lambda_3 = \frac{U_1 W^3 H'(W)}{U_4}, \quad \Lambda_4 = 2U_1 (WH'(W) + H(W)), \quad (3.8b)$$

where  $H(W)$  is an arbitrary differentiable function with

$$W := \frac{U_4}{(U_1 - U_3 + s)^{1/2}}. \quad (3.9)$$

The conserved current for system (1.5a) – (1.5d) is then

$$\Phi^t = H(w)\rho \quad (3.10a)$$

$$\Phi^x = H(w)u\rho. \quad (3.10b)$$

Here the argument,  $w$ , in the arbitrary function  $H$ , is

$$w := \frac{\rho}{(u - u_{xx} + s)^{1/2}}. \quad (3.11)$$

**Special Case 3:**  $\sigma$  arbitrary, but nonzero, with  $\sigma_1 = 1$  and  $s$  arbitrary. The integrating factors are as follows:

$$\Lambda_1 = \frac{U_3 - 2U_1 - s}{\sigma U_4}, \quad \Lambda_2 = 0, \quad (3.12a)$$

$$\Lambda_3 = \frac{U_1}{\sigma U_4}, \quad \Lambda_4 = \frac{sU_1 - \sigma U_4^2 + U_1^2 - U_1 U_3}{\sigma U_4^2}. \quad (3.12b)$$

The conserved current for system (1.5a) – (1.5d) is then

$$\Phi^t = \frac{u_{xx} - u - s}{\sigma \rho} \quad (3.13a)$$

$$\Phi^x = \frac{uu_{xx} - u^2 - \sigma \rho^2 - su}{\sigma \rho}. \quad (3.13b)$$

**Special Case 4:**  $\sigma_1 = 0$  with  $\sigma$  and  $s$  arbitrary. The integrating factors are as follows:

$$\Lambda_1 = (U_3 - 2s)H(X, W), \quad \Lambda_2 = U_2 H(X, W) \quad (3.14a)$$

$$\Lambda_3 = U_1 H(X, W), \quad \Lambda_4 = -\sigma U_4 H(X, W), \quad (3.14b)$$

where  $H$  is an arbitrary differentiable function and

$$W := \frac{1}{2} (-4sU_1 - \sigma U_4^2 + 2U_1 U_3 + U_2^2). \quad (3.15)$$

The conserved current for system (1.5a) – (1.5d) is then

$$\Phi^t = 0 \tag{3.16a}$$

$$\Phi^x = wH(t, w), \tag{3.16b}$$

where

$$w := \frac{1}{2} (-4su - \sigma\rho^2 + 2uu_{xx} + u_x^2). \tag{3.17}$$

**Special Case 5:**  $\sigma_1 = 0$  with  $\sigma$  arbitrary, but nonzero, and  $s$  arbitrary. The integrating factors are as follows:

$$\Lambda_1 = \frac{2}{\sigma} \left( sU_1 + \frac{\sigma}{2}U_4^2 - U_1U_3 \right), \quad \Lambda_2 = 0 \tag{3.18a}$$

$$\Lambda_3 = -\frac{U_1^2}{\sigma}, \quad \Lambda_4 = 2U_1U_4. \tag{3.18b}$$

The conserved current for system (1.5a) – (1.5d) is then

$$\Phi^t = \frac{\rho^2}{2} \tag{3.19a}$$

$$\Phi^x = \frac{1}{\sigma} (su^2 + \sigma u\rho^2 - u^2u_{xx}). \tag{3.19b}$$

#### 4. Integrating factors for system (1.9a) – (1.9f) and conservation laws for (1.6a) – (1.6b):

Solving conditions (2.6) and (2.7) for system (1.9a) – (1.9f), the complete set of first-order integrating factors  $\{\Lambda_1, \dots, \Lambda_6\}$ , of the form

$$\Lambda_j = \Lambda_j(X, T, U_1, \dots, U_6, U_{1,X}, \dots, U_{6,X}), \quad j = 1, 2, \dots, 6$$



are the following:

$$\begin{aligned}
 \Lambda_1 = & \lambda_1 (2U_6U_1 + 2U_3U_4 + 2U_4U_5^2 + 4U_4^2U_6 - 6U_4^3 - 2s_1U_1 - 8s_1U_4^2 - 6U_1U_4 \\
 & - 2s_2U_4 + U_{5,X}) + \lambda_2 (U_3 + 2U_5^2 - 4s_1U_4 - 3U_1 - 2s_2) \\
 & + \lambda_3 (U_6 - 3U_4 - 2s_1) + \lambda_4 (-8s_1^2U_4 - 2s_1s_2 - s_1U_1 + 2s_1U_3 - 22s_1U_4^2 \\
 & + 12s_1U_4U_6 + 2s_1U_5^2 - 4s_2U_4 + 2s_2U_6 - 3U_1U_4 + U_1U_6 + 4U_3U_4 - 2U_3U_6 - 12U_4^3 \\
 & + 14U_4^2U_6 + 4U_4U_5^2 - 4U_4U_6^2 - 2U_5U_6) Z^{-3/2} \\
 & + \frac{\lambda_5}{2} (-8s_1U_4 - 2s_2 - 3U_1 + 2U_3 - 6U_4^2 + 4U_4U_6 + 2U_5^2) Z^{-1/2} \tag{4.1a}
 \end{aligned}$$

$$\Lambda_2 = -\lambda_1 U_{4,X} + \lambda_2 U_2 + \lambda_3 U_5 \tag{4.1b}$$

$$\begin{aligned}
 \Lambda_3 = & 2\lambda_1 U_1U_4 + \lambda_2 (U_1 - 2U_4^2) + \lambda_3 U_4 \\
 & + \lambda_4 (-s_1U_1 + 8s_1U_4^2 + 2s_2U_4 + U_1U_4 + U_1U_6 - 2U_3U_4 + 6U_4^3 \\
 & - 4U_4^2U_6 - 2U_4U_5^2) Z^{-3/2} + \frac{\lambda_5}{2} U_1 Z^{-1/2} \tag{4.1c}
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_4 = & \lambda_1 (2U_1U_5^2 + 2U_1U_3 + 2U_4U_{5,X} - 2s_2U_1 - 3U_1^2 - 18U_1U_4^2 + U_{2,X} \\
 & - 16s_1U_1U_4 + 8U_1U_4U_6) + \lambda_2 (24U_4^3 - 4U_3U_4 - 4U_4U_5^2 - 12U_4^2U_6 - 2U_{5,X} \\
 & + 24s_1U_4^2 - 4s_1U_1 + 4s_2U_4) + \lambda_3 (U_3 + 4U_4U_6 - 3U_1 + 2U_5^2 - 12U_4^2 \\
 & - 12s_1U_4 - 2s_2) + 2\lambda_4 (2s_1^2U_1 - 48s_1^2U_4^2 - 20s_1s_2U_4 - 19s_1U_1U_4 - 3s_1U_1U_6 \\
 & + 20s_1U_3U_4 - 84s_1U_4^3 + 48s_1U_4^2U_6 + 20s_1U_4U_5^2 - 2s_2^2 - 5s_2U_1 + 4s_2U_3 - 18s_2U_4^2 \\
 & + 10s_2U_4U_6 + 4s_2U_5^2 - 3U_1^2 + 5U_1U_3 - 18U_1U_4^2 + 8U_1U_4U_6 + 5U_1U_5^2 + U_1U_6^2 - 2U_3^2 \\
 & + 18U_3U_4^2 - 10U_3U_4U_6 - 4U_3U_5^2 - 36U_4^4 + 42U_4^3U_6 + 18U_4^2U_5^2 - 12U_4^2U_6^2 \\
 & - 10U_4U_5^2U_6 - 2U_5^4) Z^{-3/2} + \lambda_5 U_1 (-2s_1 + 3U_4 + U_6) Z^{-1/2} \tag{4.1d}
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_5 = & \lambda_1 (4U_1U_4U_5 - U_{1,X} - 2U_4U_{4,X}) + \lambda_2 (4U_1U_5 - 4U_4^2U_5 + 2U_{4,X}) \\
 & + \lambda_3 (U_2 + 4U_4U_5) - 2\lambda_4 U_5 (s_1U_1 - 8s_1U_4^2 - 2s_2U_4 - U_1U_4 - U_1U_6 + 2U_3U_4 \\
 & - 6U_4^3 + 4U_4^2U_6 + 2U_4U_5^2) Z^{-3/2} + \lambda_5 U_1 U_5 Z^{-1/2} \tag{4.1e}
 \end{aligned}$$

$$\begin{aligned}
 \Lambda_6 = & \lambda_1 (U_1^2 + 4U_1U_4^2) - 4\lambda_2 U_4^3 + \lambda_3 (U_1 + 2U_4^2) + 2\lambda_4 (3s_1U_1U_4 + 8s_1U_4^3 + s_2U_1 \\
 & + 2s_2U_4^2 - U_1^2 - U_1U_3 + 4U_1U_4^2 - U_1U_4U_6 - U_1U_5^2 - 2U_3U_4^2 + 6U_4^4 \\
 & - 4U_4^3U_6 - 2U_4^2U_5^2) Z^{-3/2} + \lambda_5 U_1 U_4 Z^{-1/2}, \tag{4.1f}
 \end{aligned}$$

where

$$Z := s_1 U_4 - s_2 - U_1 + U_3 - 3U_4^2 + 2U_4 U_6 + U_5^2. \quad (4.2)$$

This leads to the following set of three conserved densities and conserved flux for the system (1.6a) – (1.6b):

$$\Phi_1^t = u_1 u_{0,xx} + u_1^2 u_{1,xx} - u_0 u_1 - 2s_1 u_1^2 - 2u_1^3 \quad (4.3a)$$

$$\begin{aligned} \Phi_1^x &= (u_0 + u_1^2) u_{1,xt} + 2u_0 u_1 u_{0,xx} + 2u_0 u_1 u_{1,x}^2 + (4u_0 u_1^2 + u_0^2) u_{1,xx} \\ &\quad - \frac{1}{2} u_0^2 (6u_1 + 2s_1) - u_0 (6u_1^3 + 2s_2 u_1 + 8s_1 u_1^2) - u_{0,x} u_{1,t} \end{aligned} \quad (4.3b)$$

$$\Phi_2^t = 2u_1 u_{1,xx} + u_{0,xx} - u_0 - 2u_1^2 + 2u_{1,x}^2 - 4s_1 u_1 \quad (4.4a)$$

$$\begin{aligned} \Phi_2^x &= -2u_1 u_{1,xt} + (u_0 - 2u_1^2) u_{0,xx} - 4u_1^3 u_{1,xx} + \frac{1}{2} u_{0,x}^2 + 2(u_0 - u_1^2) u_{1,x}^2 \\ &\quad - 2(s_2 + 2s_1 u_1) u_0 - \frac{3}{2} u_0^2 + 2u_1^2 (s_2 + 4s_1 u_1 + 3u_1^2) \end{aligned} \quad (4.4b)$$

$$\Phi_3^t = u_{1,xx} - u_1 \quad (4.5a)$$

$$\begin{aligned} \Phi_3^x &= (u_0 + 2u_1^2) u_{1,xx} + u_1 u_{0,xx} + u_{0,x} u_{1,x} + 2u_1 u_{1,x}^2 - (2s_1 + 3u_1) u_0 \\ &\quad - 2u_1 (s_2 + 3s_1 u_1 + 2u_1^2). \end{aligned} \quad (4.5b)$$

$$\Phi_4^t = 2(s_1 - u_{1,xx} + u_1) z^{-1/2} \quad (4.6a)$$

$$\begin{aligned} \Phi_4^x &= 2(s_1 u_0 + 8s_1 u_1^2 + 2s_2 u_1 + 3u_0 u_1 - u_0 u_{1,xx} - 2u_{0,xx} u_1 + 6u_1^3 \\ &\quad - 4u_1^2 u_{1,xx} - 2u_1 u_{1,x}^2) z^{-1/2} \end{aligned} \quad (4.6b)$$

$$\Phi_5^t = z^{1/2} \quad (4.7a)$$

$$\Phi_5^x = u_0 z^{1/2}, \quad (4.7b)$$

where

$$z := s_1 u_1 - s_2 - u_0 + u_{0,xx} - 3u_1^2 + 2u_1 u_{1,xx} + u_{1,x}^2. \quad (4.8)$$

## 5. Concluding remarks

We have derived the complete set of first-order integrating factors for the systems CH(2,1) and CH(2,2) in Cauchy-Kovalevskaya form. The corresponding sets of conservation laws related to these integrating factors have been derived for both these systems. It would certainly be interesting to calculate higher-order integrating factors, although the computations involved for such calculations appear to be rather challenging. We aim to report some results in a future paper.

We expect that the same method than was applied here could also be used to find conservation laws for more general CH-systems proposed in [11] and [14].

## Acknowledgements

ME and NE thanks the New Jersey Institute of Technology for there hospitality during their sabbatical leave at this institute. ME and NE also thank the Wenner-Gran Foundation and Luleå University of Technology for financial support.

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