Classification of integrable polynomial vector evolution equations

Vladimir V.Sokolov
Landau Institute for Theoretical Physics,
Kosygina 2, Moscow 117334, Russia
email: sokolov@landau.ac.ru

Thomas Wolf
Brock University, St.Catharines,
Ontario, Canada L2S 3A1
email: twolf@BrockU.ca

August 22, 2001

Abstract

Several classes of systems of evolution equations with one or two vector unknowns are considered. We investigate also systems with one vector and one scalar unknown. For these classes all equations having the simplest higher symmetry are listed.

1 Introduction.

The symmetry approach to the classification of integrable evolution equations (see [1, 2, 3, 4]) is proven to be the most efficient integrability test for 1+1-dimensional nonlinear PDEs.

In this paper we extend the simplest version of this approach [5, 6] to the case of so called vector evolution equations (see [7]). As an example let us consider the following vector equation

$$U_t = U_{xxx} + \langle U, U \rangle U_x, \tag{1}$$

where $U(t,x)$ is a N-component vector and $\langle\cdot,\cdot\rangle$ stands for the standard scalar product. If $N=1$ this formula reduces to the well-known modified Korteweg-de Vries (mKdV) equation $u_t = u_{xxx} + u^2 u_x$. Equation (1) belongs to a wide class [7, 8, 9, 10, 11, 12] of integrable multi-component evolution systems related to Jordan algebras and Jordan triple systems (see Appendix 1 for definitions).

S.I. Svinolupov was the first who discovered a deep relationship between some kinds of non-associative algebraic structures and integrable multi-component evolution systems. In the next two pages we formulate his results that have relations with the subject of our paper.

One of his results states that for any Jordan triple system $\{X,Y,Z\}$ defined on a vector space J the following equation

$$U_t = U_{xxx} + \{U,U,U_x\}, \tag{2}$$

where $U(x,t)$ is an unknown function with values in J is integrable (and, in particular, has higher symmetries). To derive some concrete vector integrable systems from this general result we remind that there exist two different "vector" Jordan triple systems:

$$\{X,Y,Z\} = \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle X,Z\rangle Y \tag{3}$$

and

$$\{X,Y,Z\} = \langle X,Y\rangle Z + \langle Y,Z\rangle X. \tag{4}$$

Here X,Y and Z are arbitrary N-dimensional vectors. It is clear that (3) gives rise (up to an inessential scaling parameter) to (1) while (4) yields a different integrable vector generalisation of the mKdV equation

$$U_t = U_{xxx} + \langle U,U\rangle\, U_x + \langle U,U_x\rangle\, U. \tag{5}$$

The general formula (2) produces not only one-component vector equations of mKdV-type. Let us consider the following Jordan triple system

$$\left\{ \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}, \begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix} \right\} =$$

$$\begin{pmatrix} (\langle Y^1,Z^1\rangle + \langle Y^2,Z^2\rangle)\, X^1 + (\langle X^2,Z^2\rangle - \langle X^1,Z^1\rangle)\, Y^1 + (\langle X^1,Y^1\rangle + \langle X^2,Y^2\rangle)\, Z^1 \\ (\langle Y^1,Z^1\rangle + \langle Y^2,Z^2\rangle)\, X^2 + (\langle X^1,Z^1\rangle - \langle X^2,Z^2\rangle)\, Y^2 + (\langle X^1,Y^1\rangle + \langle X^2,Y^2\rangle)\, Z^2 \end{pmatrix} \tag{6}$$

where X^1,Y^1,Z^1 are vectors of length N_1 and X^2,Y^2,Z^2 are vectors of length N_2. This Jordan triple system was not mentioned in the original papers by Svinolupov.

The corresponding system

$$U^1_t = U^1_{xxx} + \Big(\langle U^1, U^1\rangle + \langle U^2, U^2\rangle\Big)\, U^1_x + 2\langle U^2, U^2_x\rangle\, U^1$$
$$U^2_t = U^2_{xxx} + \Big(\langle U^1, U^1\rangle + \langle U^2, U^2\rangle\Big)\, U^2_x + 2\langle U^1, U^1_x\rangle\, U^2$$

with respect to vectors U^1 and U^2 of lengths N_1 and N_2 was presented in [26], although in non-explicit form this system was probably found first in [13]. Notice that if $N_1 = N_2 = 1$ then (6) is a fully decoupled system in variables $\hat{U}^1 = U^1 + U^2$ and $\hat{U}^2 = U^1 - U^2$. For $N_1 = N_2 > 1$ it is impossible to split the system by this change of variables.

One more general result by Svinolupov is related to a multi-component generalisation of the nonlinear Schrödinger equation (NLS) which can be written as a system of two equations

$$u_t = u_{xx} + u^2 v, \quad v_t = -v_{xx} - v^2 u. \tag{7}$$

It was shown in [9] that for any Jordan triple system the following system of equations

$$\begin{cases} U_t = \quad U_{xx} + \{U, V, U\}, \\ V_t = -V_{xx} - \{V, U, V\} \end{cases} \tag{8}$$

is integrable. In particular, it has higher symmetries of any order $n \in \mathbb{N}$. For the vector triple systems (3) and (4) the general formula (8) gives rise to two different integrable vector generalizations of (7):

$$\begin{cases} U_t = \quad U_{xx} + 2\langle U, V\rangle U - \langle U, U\rangle V, \\ V_t = -V_{xx} - 2\langle U, V\rangle V + \langle V, V\rangle U, \end{cases} \tag{9}$$

and

$$\begin{cases} U_t = \quad U_{xx} + \langle U, V\rangle U, \\ V_t = -V_{xx} - \langle U, V\rangle V. \end{cases} \tag{10}$$

Both systems are well known (see [14, 15]).

Finally, Jordan triple system (6) yields the following 4-vector system of NLS-type

$$\begin{cases} U^1_t = \quad U^1_{xx} + 2(\langle U^1, V^1\rangle + \langle U^2, V^2\rangle)\, U^1 + (-\langle U^1, U^1\rangle + \langle U^2, U^2\rangle)\, V^1, \\ U^2_t = \quad U^2_{xx} + 2(\langle U^1, V^1\rangle + \langle U^2, V^2\rangle)\, U^2 + (\langle U^1, U^1\rangle - \langle U^2, U^2\rangle)\, V^2, \\ V^1_t = -V^1_{xx} - 2(\langle U^1, V^1\rangle + \langle U^2, V^2\rangle)\, V^1 - (-\langle V^1, V^1\rangle + \langle V^2, V^2\rangle)\, U^1, \\ V^2_t = -V^2_{xx} - 2(\langle U^1, V^1\rangle + \langle U^2, V^2\rangle)\, V^2 - (\langle V^1, V^1\rangle - \langle V^2, V^2\rangle)\, U^2, \end{cases}, \tag{11}$$

where vectors U^1 and V^1 have length N_1 whereas U^2 and V^2 are vectors of length N_2. It is easy to see that the reduction $U^2 = V^2 = 0$ reduces the system to (9). If $N_1 = N_2$ then the reduction $U^2 = -U^1$, $V^2 = -V^1$ becomes possible and results in system (10).

For all examples presented above, the right-hand side of the equation is a homogeneous differential polynomial under a suitable weighting scheme. The differential equation

$$u_t = f(u, u_x, \dots, u_{n-1}, u_n), \qquad u_i = \frac{\partial^i u}{\partial x^i} \tag{12}$$

is said to be λ-*homogeneous* of *weight* μ if it admits the one-parameter group of scaling symmetries

$$(x,\ t,\ u) \longrightarrow (a^{-1}x,\ a^{-\mu}t,\ a^{\lambda}u).$$

For N-component systems with unknowns $u^1, ..., u^N$ the corresponding scaling group has a similar form

$$(x, t, u^1, ..., u^N) \longrightarrow (a^{-1}x,\ a^{-\mu}t,\ a^{\lambda_1}u^1, ..., a^{\lambda_N}u^N).$$

In this paper we consider the case $\lambda_1 = \dots = \lambda_N$ only.

It was proven in [16] that in the scalar case a λ-homogeneous polynomial equation with $\lambda > 0$ may possess a homogeneous polynomial higher symmetry only if

- Case 1: $\lambda = 2$;
- Case 2: $\lambda = 1$;
- Case 3: $\lambda = \frac{1}{2}$.

For example, the KdV equation $u_t = u_{xxx} + uu_x$ is homogeneous of weight 3 for $\lambda = 2$, the mKdV equation $u_t = u_{xxx} + u^2u_x$ has the weight 3 for $\lambda = 1$ and for the Ibragimov-Shabat equation

$$u_t = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x \tag{13}$$

the weight is 3 and $\lambda = \frac{1}{2}$.

According to [17, 18] equations like (13) do not exist in the matrix case. Thus the possible values of λ for matrix one-component equations possessing higher symmetries are 2 or 1.

In the paper [24] two-component systems of the form of the form

$$\begin{cases} u_t = & k_1 u_{xx} + P(u_x, v_x, u, v), \\ v_t = & k_2 v_{xx} + Q(u_x, v_x, u, v), \end{cases}$$

where $k_1 \neq k_2$, $k_i \neq 0$, have been considered. The authors assumed that P and Q are polynomials such that the system is (λ_1, λ_2)-homogeneous, where $\lambda_2 \geq \lambda_1 > 0$ and $\lambda_2 - \lambda_1$ is not a natural number. Under these assumptions it was proven that if the system possesses higher symmetries, then

- if both P and Q have no quadratic terms, then $k_2 = -k_1$;
- if $\lambda_1 = \lambda_2 = \lambda$, then $\lambda \in \{2, 1, \frac{1}{2}\}$.

Also any possible pairs of λ_1 and λ_2 were listed.

For integrable systems with more than two unknowns there are no rigorous statements describing all possible types of homogeneous polynomial systems but for all known examples with $\lambda_1 = \cdots = \lambda_N = \lambda > 0$, λ is equal to 2, 1 or $\frac{1}{2}$ just as in the scalar and two-component cases.

In this paper we systematically investigate one and two component homogeneous vector equations having higher symmetries under the assumption $\lambda \in \{2, 1, \frac{1}{2}\}$. It is remarkable that the number of different integrable cases for vector equations turns out to be greater than in the matrix case.

Methodologically our approach is not new. But the continuous increase of computer speed and development of efficient computer algebra algorithms, especially for solving overdetermined algebraic systems allows to tackle problems which were unsolvable so far.

Nevertheless, even modern technology cannot deal with multicomponent systems, if we use a straightforward component formalism. A corner stone of our approach in this paper is a special component-less representation of vectors and vector equations.

Despite of such customised data-representations the paper is the result of long and intensive computations with the help of the computer algebra package Crack [19, 20]. Although originally designed to solve overdetermined PDE-systems, it got enhanced in recent years to solve algebraic and especially bi-linear systems efficiently [21].

We find a number of new vector models supposed to be integrable. The best way to prove the integrability of these models is to find Lax representations for them. We are planning to do that in a separate paper.

2 Integrable equations with one vector unknown

In this section we consider vector equations of the form

$$U_t = f_n\, U_n + f_{n-1}\, U_{n-1} + \cdots + f_1\, U_1 + f_0\, U, \qquad U_i = \frac{\partial^i U}{\partial x^i} \tag{14}$$

with a single unknown vector-valued function U of dimension N. Here f_i are scalar functions of scalar products $\langle U_i,\, U_j\rangle$, $0 \le i \le j \le n$. We shall call equations of this form *isotropic*. It is clear that the isotropic equations are invariant with respect to any orthogonal transformation of the vector U i.e. the equation has the orthogonal group $O(N)$ as a group of point symmetries.

We shall consider equations (14) that are integrable for any dimension N. In addition, we assume that the coefficients f_i do not depend on N. In virtue of the arbitrariness of N, all different scalar products $\langle U_i,\, U_j\rangle$ with $i \le j$ can be regarded as *independent* variables. Their functional independence is a crucial requirement in all our computations. If N would be fixed, we could not assume that. For instance, if $N = 3$, then the determinant of the matrix with entries $a_{ij} = \langle U_i,\, U_j\rangle$, $i, j = 1, \ldots, 4$ identically equals to zero.

The signature of the scalar product is unessential for us. Furthermore, the assumptions that the vector space is finite-dimensional and the constant field is $\mathbb{R}$ are also unimportant. For instance, U could be a function of t, x and y and the scalar product be

$$\langle U,\, V\rangle = \int_{-\infty}^{\infty} U(t,x,y)\, V(t,x,y)\, dy.$$

In this way, our formulas and statements are valid also for this particular sort of $1+2$-dimensional non-local equations.

We suggest that the coefficients f_i are *polynomials* in $\langle U_i,\, U_j\rangle$ such that (14) is homogeneous with $\lambda = 2$ (Case 1), $\lambda = 1$ (Case 2) or $\lambda = \frac{1}{2}$ (Case 3). Given the weight of (14) it is easy to find the most general form of such an equation (we have a special symbolic code for that). Everywhere we assume that $f_n = \mathrm{const} \neq 0$.

For example, the general form for an equation of weight 3 with $\lambda = 1$ is given by

$$U_t = a_1 U_{xxx} + a_2 \langle U, U\rangle U_x + a_3 \langle U, U_x\rangle U, \tag{15}$$

where a_i are arbitrary constants, $a_1 \neq 0$. Using a scaling of t we can normalize a_1 to 1.

To describe all integrable cases we assume (cf. with the scalar case [5, 3, 17, 16]) that (15) has a higher homogeneous symmetry of weight 5, whose general form is

$$\begin{aligned} U_\tau \;\; = \;\; & U_{xxxxx} + b_1 \langle U, U\rangle\; U_{xxx} + b_2 \langle U, U_x\rangle\; U_{xx} + \\ & (b_3 \langle U, U_{xx}\rangle + b_4 \langle U_x, U_x\rangle + b_5 \langle U, U\rangle^2)\; U_x + \\ & (b_6 \langle U, U_{xxx}\rangle + b_7 \langle U_x, U_{xx}\rangle + b_8 \langle U, U\rangle \langle U, U_x\rangle)\; U. \end{aligned} \tag{16}$$

Compatibility conditions $(U_t)_\tau = (U_\tau)_t$ of (15) and (16) give rise to a system of 26 bilinear algebraic equations with respect to unknowns $a_2, a_3, b_1, \ldots, b_8$. Solving this system with the help of the symbolic computer program Crack we finally obtain that the system has a solution only if $a_3 = 0$ or $a_2 = -2a_3$. These two possibilities lead to integrable equations (1) and (5).

We considered Cases 1,2 and 3 corresponding to different weighting schemes. For each case we have performed the similar calculations to find all equations of second order having a symmetry of third order, equations of third order having a symmetry of fifth order, and equations of fifth order possessing a symmetry of seventh order. The complexity of the algebraic system increases drastically with an increase of the total weight (differential order of the equation and symmetry) and with decreasing λ. For details see Appendix 2.

It turns out that apart from equations (1) and (5), there exists only one more integrable equation:

$$U_t = U_{xxx} + 3\langle U, U\rangle U_{xx} + 6\langle U, U_x\rangle U_x + 3\langle U, U\rangle^2 U_x + 3\langle U_x, U_x\rangle U \tag{17}$$

with $\lambda = \frac{1}{2}$. This vector analogue of the Ibragimov-Shabat equation (13) was already found in [22].

All equations (1),(5), and (17) belong to infinite commutative hierarchies of evolution equations such that any equation from the hierarchy is a symmetry for all others. These three hierarchies are of the same (3,5)-type. This notation means that the simplest non-trivial equation in the hierarchy is of order three and the next is an equation of order 5.

We see, that there are no hierarchies of type (2,3) and (5,7) in the vector case with one unknown vector in contrast with the scalar case, where such hierarchies exist (Burgers, Kaup-Kupershmidt, Kupershmidt and Sawada-Kotera equations).

Combining our results with the approach by Sanders-Wang [16, 18] one can prove that any polynomial homogeneous equation of the form (14) with $\lambda > 0$ and $f_n = const \neq 0$ which has at least one higher symmetry, belongs to one of the above three hierarchies [23].

3 Integrable equations with two vector unknowns

In this section we classify integrable vector NLS-type systems. More precisely, we consider systems of the form

$$\begin{cases} U_t = U_{xx} + p_1 U_x + p_2 V_x + p_3 U + p_4 V, \\ V_t = -V_{xx} + p_5 U_x + p_6 V_x + p_7 U + p_8 V, \end{cases} \tag{18}$$

where U and V are vectors and the coefficients p_i are λ-homogeneous polynomials depending on all possible scalar products of vectors U, V, U_x, V_x. By analogy with the scalar case λ is supposed to be 2,1, or $\frac{1}{2}$. To derive the integrable cases we assume that (18), just as in the scalar case (see [2]) possesses a symmetry of the form

$$\begin{cases} U_\tau = U_{xxx} + q_1 U_{xx} + q_2 V_{xx} + q_3 U_x + q_4 V_x + q_5 U + q_6 V, \\ V_\tau = V_{xxx} + q_7 U_{xx} + q_8 V_{xx} + q_9 U_x + q_{10} V_x + q_{11} U + q_{12} V, \end{cases} \tag{19}$$

where the coefficients q_i are λ-homogeneous polynomials of all possible scalar products of $U, V, U_x, V_x, U_{xx}, V_{xx}$. We call such systems integrable. The classification result is the following.

It is easy to see that nonlinear systems (18) corresponding to the case $\lambda = 2$ do not exist.

For $\lambda = 1$ there exist only two integrable systems (10) and (9) (up to a scaling of U, V, x and t).

In the case $\lambda = \frac{1}{2}$ the complete list of integrable cases (up to the scaling and the involution $U \leftrightarrow V,\ t \leftrightarrow -t$) looks as follows:

$$\begin{cases} U_t = U_{xx} + 2\alpha\langle U,V\rangle U_x + 2\alpha\langle U,V_x\rangle U - \alpha\beta\langle U,V\rangle^2 U, \\ V_t = -V_{xx} + 2\beta\langle U,V\rangle V_x + 2\beta\langle V,U_x\rangle V + \alpha\beta\langle U,V\rangle^2 V \end{cases} \tag{20}$$

$$\begin{cases} U_t = U_{xx} + 2\alpha\langle U,V\rangle U_x + 2\beta\langle U,V_x\rangle U + \beta(\alpha-2\beta)\langle U,V\rangle^2 U, \\ V_t = -V_{xx} + 2\alpha\langle U,V\rangle V_x + 2\beta\langle V,U_x\rangle V - \beta(\alpha-2\beta)\langle U,V\rangle^2 V \end{cases} \tag{21}$$

$$\begin{cases} U_t = U_{xx} + 2\alpha\langle U,V\rangle U_x + 2\beta\langle U,V_x\rangle U + 2(\beta-\alpha)\langle V,U_x\rangle U \\ \qquad -\alpha\beta\langle U,V\rangle^2 U, \\ V_t = -V_{xx} + 2\alpha\langle U,V\rangle V_x + 2\alpha\langle V,U_x\rangle V + \alpha\beta\langle U,V\rangle^2 V \end{cases} \tag{22}$$

$$\begin{cases} U_t = U_{xx} + 2\alpha\langle U,V\rangle U_x + 2\alpha\langle U,V_x\rangle U + 2\beta\langle V,U_x\rangle U \\ \qquad -\alpha(\alpha-\beta)\langle U,V\rangle^2 U, \\ V_t = -V_{xx} + 2\alpha\langle U,V\rangle V_x + 2\alpha\langle V,U_x\rangle V + 2\beta\langle U,V_x\rangle V \\ \qquad +\alpha(\alpha-\beta)\langle U,V\rangle^2 V \end{cases} \tag{23}$$

$$\begin{cases} U_t = U_{xx} + 4\alpha\langle U,V\rangle U_x + 2(\alpha-\beta)\langle U,U\rangle V_x + 4\beta\langle U,V_x\rangle U \\ \qquad +4\beta(\alpha-2\beta)\langle U,V\rangle^2 U - 2\beta(\alpha-\beta)\langle U,U\rangle\langle V,V\rangle U \\ \qquad -4\beta(\alpha-\beta)\langle U,U\rangle\langle U,V\rangle V, \\ V_t = -V_{xx} + 4\alpha\langle U,V\rangle V_x + 2(\alpha-\beta)\langle V,V\rangle U_x + 4\beta\langle V,U_x\rangle V \\ \qquad -4\beta(\alpha-2\beta)\langle U,V\rangle^2 V + 2\beta(\alpha-\beta)\langle U,U\rangle\langle V,V\rangle V \\ \qquad +4\beta(\alpha-\beta)\langle V,V\rangle\langle U,V\rangle U \end{cases} \tag{24}$$

$$
\begin{cases}
U_t = \quad U_{xx} + 4\alpha\langle U,V\rangle U_x - 2\beta\langle U,U\rangle V_x + 4\alpha\langle U,V_x\rangle U \\
\qquad +4\beta\langle V,U_x\rangle U - 4\beta\langle U,U_x\rangle V + 6\beta(\alpha-\beta)\langle U,U\rangle\langle V,V\rangle U \\
\qquad -4\alpha(\alpha-\beta)\langle U,V\rangle^2 U - 4\beta(\alpha-\beta)\langle U,U\rangle\langle U,V\rangle V, \\
V_t = -V_{xx} + 4\alpha\langle U,V\rangle V_x - 2\beta\langle V,V\rangle U_x + 4\alpha\langle V,U_x\rangle V \\
\qquad +4\beta\langle U,V_x\rangle V - 4\beta\langle V,V_x\rangle U - 6\beta(\alpha-\beta)\langle U,U\rangle\langle V,V\rangle V \\
\qquad +4\alpha(\alpha-\beta)\langle U,V\rangle^2 V + 4\beta(\alpha-\beta)\langle V,V\rangle\langle U,V\rangle U
\end{cases}
\tag{25}
$$

All equations in the list contain arbitrary constants α and β. It is easy to see that if α is not equal to zero, then it can be reduced to 1 via scalings of t, x, U and V. Thus the essential parameter is the ratio of β and α. We choose the above form with the two arbitrary constants to avoid considering the case $\alpha = 0$ separately.

Remark 1. In the paper [25] equations (21) and (23) were found and Lax representations for them were presented. The other equations seem to be new.

Remark 2. In the paper [27] a list of matrix integrable equations have been obtained. This list contains all integrable polynomial homogeneous matrix equations with $\lambda = \frac{1}{2}$. Some particular cases of our vector systems (20)-(25) can be derived from the matrix list as reductions. For example, for all matrix systems we can perform the following reduction

$$
U = \begin{pmatrix} 0 & 0 & ... & 0 & u^1 \\ 0 & 0 & ... & 0 & u^2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & u^N \\ v^1 & v^2 & ... & v^N & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 & ... & 0 & x^1 \\ 0 & 0 & ... & 0 & x^2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & x^N \\ y^1 & y^2 & ... & y^N & 0 \end{pmatrix}
$$

to get the corresponding systems with four vector unknowns U, V, X, Y. Sometimes we can decrease the number of vector unknowns to two by further reductions and obtain some systems from the list (20)-(25). But since non of the matrix equations from [27] have arbitrary parameters and our equations do, we probably can construct only particular equations from the list (20)-(25) by reductions of the matrix equations.

4 Integrable equations with one scalar and one vector unknown

Some vector analogues of the KdV equation can be constructed using a different general result of Svinolupov [8]: for any Jordan algebra (see the definition in the

Appendix 1) with multiplication $\circ$ the following "Jordan" KdV equation

$$U_t = U_{xxx} + U \circ U_x \tag{26}$$

has higher symmetries. Since for any Jordan triple system $J = \{X, Y, Z\}$ and given fixed vector C the following multiplication

$$X \circ Y = \{X, C, Y\}$$

defines a Jordan algebra, the formulas (3) and (4) produce two vector KdV equations

$$U_t = U_{xxx} + \langle U, C\rangle\, U_x + \langle C, U_x\rangle\, U - \langle U, U_x\rangle\, C \tag{27}$$

and

$$U_t = U_{xxx} + \langle U, C\rangle\, U_x + \langle C, U_x\rangle\, U. \tag{28}$$

The first had been obtained in [7] and the second was considered in [13].

The Jordan triple system (6) gives rise to

$$\begin{cases} U^1_t = U^1_{xxx} + & (\langle U^1, C^1\rangle + \langle U^2, C^2\rangle)\, U^1_x + \\ & (\langle C^1, U^1_x\rangle + \langle C^2, U^2_x\rangle)\, U^1 + \\ & (-\langle U^1, U^1_x\rangle + \langle U^2, U^2_x\rangle)\, C^1, \\ U^2_t = U^2_{xxx} + & (\langle U^1, C^1\rangle + \langle U^2, C^2\rangle)\, U^2_x + \\ & (\langle C^1, U^1_x\rangle + \langle C^2, U^2_x\rangle)\, U^2 + \\ & (\langle U^1, U^1_x\rangle - \langle U^2, U^2_x\rangle)\, C^2, \end{cases} \tag{29}$$

Note that it is impossible to construct any vector KdV equation (with a term quadratic in U) without using constant vectors. Indeed, the simplest nonlinear vector term which can be generated with the help of the scalar product from only U, U_x, U_{xx} is of third degree.

For non-isotropic equations like (28) one can use the orthogonal group $O(N)$ to simplify the vector C and bring it, for example, to the form

$$C = (1, 0, 0, ..., 0). \tag{30}$$

With vector C fixed, the equation admits the symmetry group $O(N-1)$.

The component form of (28) with C given by (30) is the following

$$\begin{cases} u^1_t = u^1_{xxx} + 2u^1 u^1_x, \\ u^2_t = u^2_{xxx} + u^1 u^2_x + u^1_x u^2, \\ \dots \\ u^N_t = u^N_{xxx} + u^1 u^N_x + u^1_x u^N, \end{cases}$$

where $U = (u^1, \dots, u^N)$. We see that the first component u^1 satisfies the standard scalar KdV equation and the remaining equations are linear if the function u^1 is already found. We call such vector systems *triangular*.

In contrast, (27) cannot be splited by any orthogonal transformation, i.e. it is *non-triangular*. Note that if C is of the form (30) then (27) can be rewritten as

$$\begin{cases} u_t = u_{xxx} + uu_x - \langle U, U_x\rangle, \\ U_t = U_{xxx} + uU_x + u_x U, \end{cases} \tag{31}$$

where $u = u^1$ and $U = (u^2, \dots, u^N)$.

System (31) gives us an example of an integrable systems with one vector and one scalar unknown. Surprisingly, for such systems the list of integrable models seems to be much richer than in the case of isotropic equations. In the following a lower case u stands for the scalar variable and the vector variable is denoted by a capital U.

4.1 The case $\lambda = 2$

In this paper we consider the simplest case $\lambda = 2$. The generic form for a second order homogeneous system of this type reads as follows

$$\begin{cases} u_t &= a_1\, u_{xx} + a_2\, u^2 + a_3\, \langle U,\, U\rangle, \\ U_t &= a_4\, U_{xx} + a_5\, u\, U, \end{cases} \tag{32}$$

where a_i are constants. We assume that at least one of the constants a_1 or a_4 is non-zero. It means that the system is non-degenerate. As in the scalar case, the classification result is negative. It turns out that system (32) has a symmetry of third order only if $a_2 = a_3 = a_5 = 0$.

For third order systems with $\lambda = 2$ the result is much more interesting. The generic form of such systems is given by

$$\begin{cases} u_t &= a_1\, u_{xxx} + a_2\, uu_x + a_3\, \langle U,\, U_x\rangle, \\ U_t &= a_4\, U_{xxx} + a_5\, u\, U_x + a_6\, u_x\, U. \end{cases} \tag{33}$$

These systems can be regarded as natural vector generalizations of the KdV-equation. We assume that 1) the system is of third order, i.e. at least one of the constants a_1 or a_4 is non-zero; 2) our system is non-triangular, i.e. $a_3 \neq 0$ and one of a_5 or a_6 is non-zero.

Theorem. Suppose a non-triangular system (33) possesses a symmetry of fifth order. Then (up to a scaling of t, x, u and U) this system coincides with (31), or belongs to the following list

$$\left\{ \begin{array}{rcl} u_t & = & u_{xxx} + 3uu_x + 3\langle U, U_x\rangle, \\ U_t & = & u\,U_x + u_x\,U, \end{array} \right. \tag{34}$$

$$\left\{ \begin{array}{rcl} u_t & = & \langle U,\, U_x\rangle, \\ U_t & = & U_{xxx} + 2u\,U_x + u_x\,U, \end{array} \right. \tag{35}$$

$$\left\{ \begin{array}{rcl} u_t & = & u_{xxx} + uu_x + \langle U, U_x\rangle, \\ U_t & = & -2\,U_{xxx} - u\,U_x. \end{array} \right. \tag{36}$$

System (34) has been considered by Kupershmidt [28]. It is a vector generalisation of the Ito equation.

Systems (35) and (36) seem to be new. They are vector generalisations of the corresponding scalar systems from the paper [29]. A coherent picture of such vector analogues of the Kac-Moody KdV-systems will be developed in a separate paper. It is interesting to note that the fifth order symmetry

$$\begin{array}{rcl} u_t & = & u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x - \\ & & 10\langle U, U_{xxx}\rangle - 15\langle U_x, U_{xx}\rangle - 10u_x\langle U, U\rangle - 20u\langle U, U_x\rangle, \\ U_t & = & -9\,U_{xxxxx} - 30u\,U_{xxx} - 45u_x\,U_{xx} - (35u_{xx} + 20u^2 + 5\langle U, U\rangle)\,U_x \\ & & -(10u_{xxx} + 20uu_x + 5\langle U, U_x\rangle)\,U \end{array} \tag{37}$$

of system (35) is nothing but a vector generalization of the Kaup-Kuperschmidt equation. Indeed, if the vector part is absent (i.e. $U = 0$) then system (35) becomes trivial and (37) turns out to be the Kaup-Kuperschmidt equation.

Our attempts to find vector analogues for the Sawada-Kotera equation among systems with one vector and one scalar unknowns were unsuccessful. It turns out that any non-triangular, non-degenerate system of fifth order with $\lambda = 2$, having a symmetry of seventh order, is a symmetry of one of (31), (34), (35) or (36).

Appendix 1.

Definition of Jordan algebra: A vector space J equipped with a product $\circ : J \times J \to J$ such that

$$X \circ Y = Y \circ X, \qquad X^2 \circ (Y \circ X) = (X^2 \circ Y) \circ X$$

for any $X, Y \in J$ is called *Jordan algebra.*

If $*$ is a multiplication in an associative algebra then $X \circ Y = X * Y + Y * X$ is a Jordan product.

Definition of Jordan triple system: A vector space J equipped with a triple product $\{\cdot,\cdot,\cdot\} : J \times J \times J \to J$ such that

$$\{X, Y, Z\} = \{Z, Y, X\}$$

$$\{X, Y, \{V, W, Z\}\} - \{V, W, \{X, Y, Z\}\} = \{\{X, Y, V\}, W, Z\} - \{V, \{Y, X, W\}, Z\}$$

for any $V, W, X, Y, Z \in J$ is called *Jordan triple system.*

Any associative algebra with a multiplication $*$ produces a Jordan triple system with respect to the triple product

$$\{X, Y, Z\} = X * Y * Z + Z * Y * X.$$

Appendix 2.

For different ansätze of a vector equation and its symmetry, like equations (15) and (16), the algebraic conditions for the unknown coefficients vary drastically in their size. For the nine combinations of 3 values $\lambda = 2, 1, \frac{1}{2}$ and 3 pairs of differential order of the equation and its symmetry ((2+3),(3+5),(5+7)) the table below shows the number of unknowns, the number of conditions and the total number of terms in these conditions for the resulting nine over-determined systems of algebraic conditions.

order $\to$	2 + 3	3 + 5	5 + 7	
$\lambda = 2$	0	2	8	# of unknowns
	0	4	8	# of equations
	0	4	8	total # of terms
$\lambda = 1$	3	10	31	# of unknowns
	5	26	198	# of equations
	9	121	3125	total # of terms
$\lambda = \frac{1}{2}$	10	33	107	# of unknowns
	21	129	927	# of equations
	80	1603	52677	total # of terms

Table 1. A comparison of the size of the symmetry conditions for single vector equations.

Acknowledgement

The authors are grateful to Jan Sanders and Jing Ping Wang for useful discussions. They are also thankful to Takayuki Tsuchida for comments about the integrability of systems found. The first author (V.S.) was supported, in part, by RFBR grant 99-01-00294, INTAS grant 99-1782, and EPSRC grant GR K99015. The second author (T.W.) thanks Winfried Neun for computer algebra related discussions and the UMS MEDICIS service at the CNRS/Polytechniqe in computer algebra for providing access to their computing facilities. Both authors thank the Newton Institute, Cambridge for its hospitality during the programme "What is Integrability" where the manuscript was completed.

References

[1] Sokolov V.V. and Shabat A.B., Classification of integrable evolution equations Soviet Sci. Reviews, Section C, **4** 221-280 (1984).

[2] Mikhailov A.V., Shabat A.B., and Yamilov R.I., The symmetry approach to the classification of non-linear equations. Complete lists of integrable systems, *Russian Math. Surveys*, **42**(4), 1–63, 1987.

[3] Fokas A.S., Symmetries and integrability, *Stud. Appl. Math.*, **77**, 253–299, 1987.

[4] Mikhailov A.V., Shabat A.B., Sokolov V.V., Symmetry Approach to Classification of Integrable Equations, in the book "What is integrability?", Springer-Verlag (Springer Series in Nonlinear Dynamics), 115-184 (1991).

[5] N.Kh. Ibragimov and A.B. Shabat, Evolutionary equations with nontrivial Lie-Bäcklund algebra. Func. Anal. Appl., **14(1)** 19-28 (1980).

[6] Fokas A.S., A symmetry approach to exactly solvable evolution equations, J. Math. Phys., **21(6)** 1318-1325 (1980).

[7] Sokolov V.V., Svinolupov S.I., Vector-matrix generalizations of classical integrable equations, Theor. Math. Phys., **100(2)** 959-962 (1994).

[8] Svinolupov S.I., Jordan algebras and generalized KdV equations, Teor. i Mat. Fiz., **87(3)** 391-403 (1991).

[9] Svinolupov S.I., Generalized Schrödinger equations and Jordan pairs, Comm. in Math. Physics, **143** 559-575 (1992).

[10] Svinolupov S.I., Jordan algebras and integrable systems, Func. analiz i pril., **27(4)** 40-53 (1993).

[11] Sokolov V.V., Svinolupov S.I., Deformations of Jordan triple systems and integrable equations, Theor. Math. Phys., **108(3)** 1160-1163 (1996).

[12] Habibullin I.T., Sokolov V.V., Yamilov R.I., Multi-component integrable systems and nonassociative structures, in "Nonlinear Physics: theory and experiment, **World Scientific Publishing** 139-168 (1995).

[13] Athorne C., Fordy A.P., Generalised KdV and MKdV equations associated with symmetric spaces, J. Phys.A: Math. Gen., **20** 1377-1386 (1987).

[14] Kulish P.P. and Sklyanin E.K., O(N)-invariant nonlinear Schrodinger equation. - A new completely integrable system. Phys. Lett. A, 84(7), 349-352, 1981

[15] Manakov S.V., On theory of two-dimensional self-focussing of electromagnetic waves, ZhETF, **65** 505-516 (1973).

[16] Sanders J.A., Wang J.P., On the integrability of homogeneous scalar evolution equations. J. Diff. Eq., **147** 410–434 (1998).

[17] Olver P.J., Sokolov V.V., Integrable Evolution Equations on Associative Algebras, Commun. in Math. Phys., **193** 245-268 (1998).

[18] Olver P.J., Wang J.P, Classification of Integrable One-Component Systems on Associative Algebras, Proc. London Math. Soc., **81(3)** 566–586 (2000).

[19] Wolf, T., Brand, A., The Computer Algebra Package CRACK for Investigating PDEs, Package in the computer algebra system REDUCE. Editors: Anthony C. Hearn and John P. Fitch: REDUCE User's Manual 3.7, Konrad–Zuse–Zentrum, (1999).

[20] Wolf, T., The program CRACK for solving PDEs in General Relativity, in Relativity and Scientific Computing, Eds. F.W.Hehl, R.A.Puntigam, H.Ruder, Springer, 241–257 (1996).

[21] Wolf, T., Size reduction and partial decoupling of systems of equations, QMW-preprint, University of London (1999).

[22] Sokolov V.V., Wolf T., A symmetry test for quasilinear coupled systems, Inverse Problems, **15** L5-L11 (1999).

[23] Sanders J.A., Wang J.P., Private communication.

[24] Sanders J.A., Wang J.P., On the integrability of systems of second order evolution equations with two components. submited to J. Diff. Eq., ().

[25] Tsuchida T., Wadati M., New integrable systems of derivative nonlinear Schrödinger equations with multiple components, Phys. Lett. A, **257** 53–64 (1999).

[26] Sakovich S.Yu., Tsuchida T., Symmetrically coupled higher-order nonlinear Schrödinger equations: singularity analysis and integrability, J. Phys. A: Math. Gen., **33** 7217–7226 (2000).

[27] Olver P.J., Sokolov V.V., Non-abelian integrable systems of the derivative nonlinear Schrödinger type, Inverse Problems, **14(6)** L5-L8 (1998).

[28] Kupershmidt B.A., A coupled Korteweg-de Vries equation with dispersion, J. Phys. A, **18** L571-L573 (1985).

[29] Drinfeld, V.G. and Sokolov, V.V., Lie algebras and equations of Korteweg de Vries type, Jour. Sov. Math., **30** 1975-2036 (1985).